Some Results on the Equivariant Ginzburg-Landau Vortex in Estimate Dimension

بعض النتائج على دوامة جينزبورغ – لانداو المكافئة في البعد التقديري

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المستخلص

رمزنا بالدالة $O(N)$ للدوامة المكافئة لحل نوع معادلات جينزبورغ – لانداو ذات البعد النوني لفضاء إقليدس ، برهنا الحد الأدنى من الطاقة المحلية لدالة الطاقة المقابلة ، توصلنا إلى بعض النتائج على المكافئ في البعد التقديري .

Abstract

We characterize the $O(N)$-equivariant vortex solution for Ginzburg-Landau type equations in the N-dimensional for Euclidean space and we prove its local energy minimality for the corresponding energy functional. We concluded some results on the equivariant in estimate dimensions.

1. Introduction:

Adriano Pisante continue the study of energy minimalist property for maps $u : R^N \to R^N$ which are entire (smooth) solutions of the system

$$\Delta u + u(1 - |u|^2 = 0 \qquad (1.1)$$

in dimension $N \geq 3$ [22]. The case $N = 3$ has been extensively treated in [18] in the spirit of theimportant work [19] concerning the case $N = 2$ which is the truly relevant one in the studyof vortices in Ginsburg–Landau theory of superconductivity (see e.g. [3,20] and referencestherein).

The system (1.1) is naturally associated to the energy functional

$$E(v,\Omega) \int_\Omega \left(\frac{1}{2}|\nabla_v| + \frac{1}{4}(1 - |v|^2)^2\right) (1.2)$$

defined for $v \in X := H^1_{loc}(R^N; R^N) \cap L_4^{loc}(R^N; R^N)$ and a bounded open set $\Omega \subset R^N$. Indeed,if $u \in X$ is a critical point of $E(\cdot, \Omega)$ for every Ω then u is a weak solution of (1.1) and thus aclassical solution according to the standard regularity theory for elliptic equations. In addition,any weak solution $u \in X$ of (1.1) satisfies the natural bound $|u| \geq 1$ in the entire space, see [10,Proposition 1.9].

A natural “boundary condition” at infinity, namely

$$|u(x)| \to 1 \text{ as } |x| \to +\infty \qquad (1.3)$$

is usually added to rule out solutions with values in a lower dimensional Euclidean space and tosingle out genuinely *N*-dimensional solutions of (1.1) with nontrivial topology at infinity.More precisely, under the assumption (1.3) the map *u* has a well-defined topological degreeat infinity given by

$$\deg_{\infty} u := \deg\left(\frac{|u|}{|u|}, \partial B_R\right)$$

whenever *R* is large enough, and we are interested in solutions satisfying deg∞ *u* = 0.

A special symmetric solution *U* to (1.1) has been constructed in [1] and [13] in the form

$$U(x) = \frac{x}{|x|}, f(|x|), \tag{1.4}$$

for a unique function f vanishing at zero and increasing to one at infinity. Actually, the map U given by (1.4) is the unique $O(N)$ equivariant solution of (1.1), i.e. $U(T\,x) \equiv T\,U(x)$ for any $T \in O(N)$ (see [13]). Taking into account the obvious invariance properties of (1.1) and (1.2),

infinitely many solutions can be obtained from (1.4) by translations on the domain and orthogonal transformations on the image. In addition, these solutions satisfy $R^{2-N}E(U, B_R) \to \frac{1}{2}\frac{N-1}{N-2}|s^{N-1}|$ as $R \to +\infty$, so that *U* has infinite energy in R*N* . It is also easyto check that *U* as in (1.4) satisfies $|U_{(x)}| = 1 + O(|x|^{-2})\ as\ |x| \to +\infty$ and $deg_{\infty}\,U = 1$. In [4], H. Brezis has formulated the following very natural problem:

Is any solution u to (1.1) satisfying (1.3) (possibly with a “good” rate of convergence) and $\deg_{\infty} u = \pm 1$, of the form (1.4) (up to a translation on the domain and an orthogonal transformation on the image)?

The answer to the previous problem is affirmative when $N = 3$, see [18], at least under the assumption $|u(x)| = 1 + O(|x| - 2)\ as\ |x| \to +\infty$. In higher dimension the answer turns out to benegative in general. Indeed, following [1] it is possible to look for solutions of (1.1) in the form

$$u(x) = \omega\left(\frac{x}{|x|}\right) f(|x|), \tag{1.5}$$

for suitable harmonic maps $\omega \in C^{\infty}(S^{N-1}; S^{N-1})$ with constant energy density on *SN*-1 (thisconstant being an eigen valus of the Laplace–Beltrami operator on the sphere and the componentsof the maps being in turn corresponding Eigen functions) and for suitable profile functions $f \in C^2(R_+)$ increasing from zero to one (depending only on this constant density). At least for *N* = 8a solution of (1.1) in the form (1.5) has been constructed in [11] with degree one at infinityfor aharmonic map *ω* different from the identity.

However, if we add a further assumption on the energy growth at infinity then the previousproblem has a positive answer. Indeed we have the following characterization of the equivariantvortex solution (1.4).

Theorem 1.1. Let $N \geq 3$ and let $u \in X$ be an entire solution of (1.1). The following areequivalent:

(i) $u$satisfies:$|u(x)| \to 1$ as $|x| \to +\infty, deg_\infty u = \pm 1$ and $E(u, B_R) = \frac{1}{2}\frac{N-1}{N-2}|S^{N-1}|R^{N-2} + o(R^{N-2})$ as $R \to \infty$;

(ii) *up to a translation on the domain and an orthogonal transformation on the image, u*is$O(N)$-*equivariant, i.e.,* $u = U$*as given by* (1.4).

The previous characterization of the equivariant solution relies on the division trick introduced in [19] and a suitable improvement of the integral identity used in [18] in the case$N = 3$. As a consequence, the result in [18] extends to every dimension but no precise behavior of thesolution at infinity is needed in the proof except its energy growth at infinity. Note that, theassumptions on the modulus and the degree are only used to infer that *u* vanish at some point,which readily gives the translation parameter in the final formula.

In the three dimensional situation a more precise characterization of (1.4) was given in [18]in terms of local energy minimalist according to the following general definition.

Definition1.2. A map $\boldsymbol{u} \in \boldsymbol{X} := \boldsymbol{H}^1_{loc}(R^N; R^N) \cap \boldsymbol{L}^4_{loc}(R^N; R^N)$ is a local minimizer of *E(·)* if

$$E(u, \Omega) \leq E(v, \Omega) \quad (1.6)$$

for any bounded open set $\Omega \subset \mathbb{R}^N$and for every $v \in X$satisfying $v - u \in H^1_0(\Omega; R^N)$.Obviously local minimizers are smooth entire solutions of (1.1) but it is not clear that for each $N \geq 3$no constant local minimizers do exist or if the solutions obtained from (1.4) are locally minimizing. The main goal of this paper is to discuss local minimalists in the sense of the definition above for the solutions given by (1.4) in any dimension $N \geq 3$. Following ideas introduced in [18] in the three dimensional case, first we show existence of a no constant local minimizers$u$ vanishing at the origin and satisfying the correct energygrowth at infinity (see Theorem 3.4 for the precise statement) and then, arguing as in the proofof Theorem 1.1 we show its symmetry, i.e. we show that $u$is given by (1.4).

The existence of a no constant local minimizes of $E(\cdot)$is ultimately related to the minimalityand uniqueness property of $u_\infty(x) = \frac{x}{|x|}$for the Dirichlet integral on the unit ball among mapsin $H^1_{Id}(R^N; S^{N-1})$, which makes a strong connection of our problem with the theory of minimizing harmonic maps. These two properties of u_∞ are well known for $N = 3$ (see [5]) and for$N \geq 7$ (see [14] and [2] respectively), see also [22]. Some years later a striking simple proof of the minimality.property of u_∞ was given in [15] for any $N \geq 3$ and then uniqueness follows

The construction of a nonconstant local minimizers relies indeed on the analysis of the vorticityset for solutions $u_\lambda$to

$$(P_\lambda)\begin{cases}\Delta u+\lambda^2 u\,(1-|u|^2) &= 0 \quad in\ B_1, \\ u &= \boldsymbol{Id} \quad \boldsymbol{on}\ \partial B_1,\end{cases} \quad \lambda > \mathbf{0}, \mathbf{(1.7)}$$

which are absolute minimizers of the Ginzburg–Landau functional $E_\lambda(u, B_1)$on $H^1_{Id}\ (B_1; \mathbb{R}^N)$ where

$$E_\lambda(u,\Omega) := \int_\Omega e_\lambda\,(u)dx \quad with\ e_\lambda(u): \frac{1}{2}|\nabla u|^2 + \frac{\lambda^2}{4}(1-|u|^2)^2$$

We will show that $u_\lambda \to u_\infty\ in\ H^1\ (B_I; \mathbb{R}^N)\ as\ \lambda \to \infty$, so that the zeros of $u_\lambda$will tend to the origin. Thus, up to translations, we will obtain a locally minimizing solution to (1.1) as alimit of $u_{\lambda_n}(x/\lambda_n)$for some sequence $\lambda_n \to +\infty$. In addition, the correct energy bound, namely $E(u, B_R)\ \frac{1}{2}\frac{N-1}{N-2}\ |S^{N-1}|R^{N-2}$ for all R >0, will follow from the explicit boundary condition in (1.7) which gives the bound $E_\lambda(u_\lambda, B_1) \leq \frac{1}{2}\frac{N-1}{N-2}\ |S^{N-1}|$ and the following celebrated monotonicity formula proved in [17].

Lemma 1.3 (Monotonicity formula). Assume that $u:\ \Omega \to \mathbb{R}^N$ is a smooth solution of the systemu $+\ \lambda^2 u(1-|u|^2) = 0$in some open set $\Omega \subset \mathbb{R}^N$ and $\lambda > 0$. Then,

$$\frac{1}{R^{N-2}}E_\lambda(u, B_R(x_0)) = \frac{1}{r^{N-2}}E_\lambda(u, B_r(x_0)) + \int_{B_R(x_0)\setminus B_r(x_0)} \frac{1}{|x-x_0|}\left|\frac{\partial_u}{\partial_{|x-x_0|}}\right| dx$$

$$+\frac{\lambda^2}{2}\int_r^R \frac{1}{t^{N-2}}\int_{B_t(x_0)}(1-|u|^2)^2\,dx\,dt, \tag{1.8}$$

for $any\ x_0 \in \Omega\ and\ any\ 0 \leq r \leq R \leq\ dist(x_0, \partial\Omega)$.

As already outlined above, once we have a local energy minimizer vanishing at the origin andwith the correct bound on the energy at infinity, we can argue as in the proof of Theorem 1.1 andwe obtain the main result of the paper.

Corollary1.4. [22]: *Let* $N = 3 + \varepsilon, \varepsilon \in \mathbb{N}$and let U be the solution of (1.1) given by (1.4). Then U is a local minimizer of the energy E according to Definition 1.2. In particular, U is stable and the followinginequality holds for any bounded open set$\Omega \subset \mathbb{R}^{3+\varepsilon}$and for any $\varphi \in C_0^\infty(\Omega; \mathbb{R}^{3+\varepsilon})$,

$$\int |\nabla_\varphi|^2 + (|U|^2 - 1)|\varphi|^2 + 2\,|\boldsymbol{U}.\varphi|^2\,dx \geq 0. \tag{1.9}$$

The stability inequality was already known. Indeed, in [13] a direct stability analysis for thelinearized operator at $U$was performed in any dimension$N = 3 + \varepsilon, \varepsilon \in \mathbb{N}$, in the same spirit of the twodimensional result in [9], using block diagonalization and Perron–Frobenius type arguments.Here, instead, inequality (1.9) is obtained as a straightforward consequence of a much deeperproperty of U, namely the local energy minimality property given in Definition 1.2, with respectto arbitrarily large (but compactly supported) perturbation.

Finally, note that both Theorem 1.1 and Corollary 1.4 also apply to the case $N = 3$, which was essentially covered in [18]. Here, however, the proofs are much simpler and do not rely neither on the deep concentration-compactness and quantization results in [17,16], nor on a preciseasymptotic analysis at infinity inspired to the one for harmonic maps at isolated singularitiesgiven in [21], which was an important ingredient in [18].

The plan of the paper is the following. In Section 2 we review the properties of the equivariantsolution (1.4) and we prove Theorem 1.1. In Section 3 we study minimizing solutions(P_λ), weprove Theorem 3.4 and the main result of the paper.

2. A characterization of the EquivariantSolution:

In this section first we collect some preliminary results about the equivariant entire solution (1.4) and then we prove its characterization in terms of topological degree and asymptoticgrowth rate of the energy at infinity.

The existence and uniqueness statement and the qualitative study of the profile function $f$in (1.4) are essentially contained in [1,12,13]. In the following lemma we stress the asymptoticbehaviour at infinity. The proof is exactly the same as in [18] and will be omitted

Lemma 2.1. There is a unique solution$f \in C^2([0,+\infty))$ *of*

$$\begin{cases} f'' + \frac{2+\varepsilon}{r} f' - \frac{N-1}{r^2} f + f(1 - f^2) = 0 \\ f(0) = 0 \qquad and \quad f(+\infty) = 1. \end{cases} \tag{2.1}$$

$$In\ addition, 0 < f(r) < 1\ for\ each\ r > 0,\ f'(0) > 0, f\ is\ strictly\ increasing$$

$$R^2|f''(R)| + R\, f'(R)|N-1-R^2(1-F(R)^2)| = o(1)\ \ as\ R \to +\infty, \tag{2.2}$$

and

$$\frac{1}{R^{N-2}} \int_0^R \left(\frac{r^2}{2}(f')^2 + \frac{N-1}{2} + r^2\frac{(1-f^2)^2}{4}\right) r^{N-3}\, dr \to \frac{1}{2}\frac{N-1}{N-2}\ as\ R \to +\infty\,. \tag{2.3}$$

A straightforward consequence of the previous lemma is the following result.

Proposition 2.2. *Let* $x_0 \in \mathbb{R}^N$ *and* $T \in O(N)$. *Consider the function* $f : [0,+\infty) \to [0,1)$givenby Lemma 2.1 and define

$$w(x) := \frac{T(x-x_0)}{|x-x_0|} f(x-x_0)\,.$$

Then w is a smooth solution of (1.1). In addition, $0 < |w(x)| < 1$ for each $x \neq x_0$, w satisfies $|w(x)| = 1 + O(|x|^{-2})$as $|x| \to \infty$, $\deg_\infty w = \det T = \pm 1$and

$$\lim_{R\to+\infty} \frac{1}{R^{N-2}} \int_{B_R(x_0)}^{R} \left(\frac{1}{2}|\nabla \boldsymbol{w}(x)|^2 + \frac{(1-|\boldsymbol{w}(x)|^2)^2}{4}\right) dx\ \frac{1}{2}\frac{N-1}{N-2}\left|\boldsymbol{S}^{N-1}\right| \tag{2.4}$$

Proof. As in [1] and [13], w is smooth and it is a classical solution of (1.1) and clearly $|w(x)| \to 1\ as\ |x| \to \infty, deg_{\infty}w = det\ T$. Finally, a simple calculation yield

$$\frac{1}{2}|\nabla \boldsymbol{w}(\mathrm{x})|^2 + \frac{\left(1-\left|\boldsymbol{w}(\mathrm{x})\right|^2\right)^2}{4} = \frac{1}{2}\left(f'(\boldsymbol{x}-\boldsymbol{x_0})\right)^2 + \frac{(N-1)}{2}\frac{\left(f'(\boldsymbol{x}-\boldsymbol{x_0})\right)^2}{|\boldsymbol{x}-\boldsymbol{x_0}|^2} + \frac{\left(1-\left|f(|\boldsymbol{x}-\boldsymbol{x_0}|^2)\right|^2\right)}{4} \tag{2.5}$$

whence (2.4) follows easily from (2.3).

Remark 2.3. Note that, in view of (2.2) and (2.5), the function $w(x)$above also satisfies thecondition $\frac{1}{2}|\nabla w(x)|^2 + \frac{(1-|w(x)\,|^2)^2}{4} = \frac{N-1}{2}\frac{1}{|x|^2} + o(|x|^{-2})\ as\ |x| \to \infty$for any $\boldsymbol{x_0} \in \mathbb{R}^N$, whence$\mathrm{E(w}, \mathrm{B}_R) = \frac{1}{2}\frac{N-1}{N-2}|\boldsymbol{S^{N-1}}| + (\boldsymbol{R^{N-2}})\ \boldsymbol{as}\ \mathrm{R} \to \infty$.

The main ingredient in the proof of Theorem 1.1 is given by the following auxiliary resultwhich is of independent interest and will be used also in the next section.

Proposition 2.4 .*Let* $u \in C^2(\mathbb{R}^N; \mathbb{R}^N)$an entire solution of (1.1) and suppose that$u(0) = 0$*and* $E(u, \mathrm{B}_R) \le \frac{1}{2}\frac{N-1}{N-2}|\boldsymbol{S^{N-1}}|$*for each* $R > 0$. *Then, there exists* $T \in O(N)$*such that*$u(x) = T\,U(x)$, *where* U*is given by* (1.4).

Proof. First we apply the division trick of [19] to prove that u has the form (1.5) with the function$f$ as in Lemma 2.1. Then a simple argument calculating the energy at infinity willgive the conclusion.*Let* $f \in C^2([0, \infty))$given by Lemma 2.1 and define

$$v(x) := \frac{u(x)}{f(|x|)} \tag{2.6}$$

The following lemma gives the basic properties of the function v that we need in the sequel.

Lemma 2.5. *Let* v*as defined in* (2.6). *Then* $v \in C^2(\mathbb{R}^N \setminus \{0\}; \mathbb{R}^N)$,

$$v(x) = B\frac{x}{|x|} + o(1)\ and\ \nabla_v(x) = \nabla\left(B\frac{x}{|x|}\right) + o(|x|^{-1}), where\ B := \frac{\nabla_u(0)}{f'(0)}, \tag{2.7}$$

$as|x| \to 0\ and\ \ finally$

$$\lim_{R\to\infty}\frac{1}{R^{N-2}}E(v, B_R) \le \frac{1}{2}\frac{N-1}{N-2}|\boldsymbol{S^{N-1}}|,$$

$$\lim_{R\to\infty}\int_{B_R}\frac{1}{R^{N-2}}\frac{(N-|v|^2)}{|x|^2}dx = 0, \tag{2.8}$$

Proof. Since u is smooth the same holds for v outside the origin and (2.7) follows easily from Taylor expansion of u near the origin. In order to prove (2.8) is suffices to show that

$$\int_{B_R} \frac{(1-|v|^2)}{4} dx = 0(R^{N-2}) = \int_{B_R} \frac{(1-|u|^2)}{4} dx,$$

and

$$\lim_{R\to\infty} \frac{1}{R^{N-2}}, \int_{B_R} \frac{1}{2}|\nabla v|^2 dx \ \lim_{R\to\infty} \frac{1}{R^{N-2}} \int_{B_R} \frac{1}{2}|\nabla u|^2 dx,$$

as $R \to \infty$, where the last limit exists because of the monotonicity formula (1.8). Indeed, (2.8)follows easily from the two equalities above combining the definition of E, the energy growth of$u$ at infinity and Young's inequality. To prove the first statement above, it is enough to note that bydefinition $|1-|v|^2| \leq f^{-2}(1-f^2 + |1-|u|^2|)$when $|x| \geq 1$. Thus, the claim on the potential partof the energy follows easily from Young's inequality and the corresponding property for u (thelatter being a simple consequence of the monotonicity formula exactly as in that $f(|x|) = 1 + O(|x|^{-2})$ *and* $f'(|x|) = o(|x|^{-2})$infinity. Since (2.2) yields

$$|\nabla v|^2 = \frac{1}{f^2}|\nabla v|^2 + |u|^2\left(\frac{f'}{f}\right)^2 - \frac{f'}{f^3}\frac{\partial}{\partial r}|u|^2 = \left(1+o(1)\right)|\nabla u|^2 + (|x|^{-2})$$

as $|x| \to \infty$, the conclusion follows by integration and straightforward manipulations.

As $u$solves (1.1) and $f$solves (2.1), simple computations lead to

$$\Delta v. f^2 v\,(1-|v|^2) = -2\frac{f'}{f}\frac{x}{|x|}.\nabla v - \frac{N-1}{|x|^2}\,v. \tag{2.9}$$

On the other hand, as $r^{3-N}\frac{\partial v}{\partial r} = \frac{x}{|x|^{N-2}}\nabla v$straightforward calculations give

$$\Delta v. r^{3-N}\frac{\partial v}{\partial r} = \frac{N-2}{|x|^{N-2}}\left|\frac{\partial v}{\partial r}\right|^2 + div\left(-\frac{1}{2}|\nabla v|^2\frac{x}{|x|^{N-2}} + \frac{1}{|x|^{N-3}}\nabla v.\frac{\partial v}{\partial r}\right)$$

$$f^2 v(1+|v|^2).r^{3-N}\frac{\partial v}{\partial r} = \left(\frac{(1+|v|^2)}{4}\right)\left(2\frac{f'f}{|x|^{N-3}} + \frac{2f^2}{|x|^{N-2}}\right)$$

$$-div\left(\frac{x}{|x|^{N-2}}f^2\frac{(1+|v|^2)^2}{4}\right)$$

$$-\frac{N-1}{|x|^2}\,v.r^{3-N}\frac{\partial v}{\partial r} = div\left(\frac{N-1}{2}\frac{x}{|x|^N}(1-|v|^2)\right)$$

Thus, multiplying Eq. (2.9) by $r^{3-N}\frac{\partial v}{\partial r}$and taking the previous identities into account yields

$$0 \leq G(x); = \left|\frac{\partial v}{\partial r}\right|^2\left(\frac{N-2}{|x|^{N-1}} + 2\frac{f'}{f}\frac{1}{|x|^{N-3}}\right) + \left(\frac{(1-|v|^2)^2}{4}\right)\left(2\frac{f'f}{|x|^{N-3}} + \frac{2f^2}{|x|^{N-2}}\right)$$

$$= div\, \boldsymbol{\Phi}(\boldsymbol{x}),(2.10)$$

Where

$$\boldsymbol{\Phi}(\boldsymbol{x}); = \frac{\mathbf{1}}{\mathbf{2}}|\nabla u|^2 \frac{x}{|x|^{N-2}} - \frac{1}{|x|^{N-3}}\nabla v.\frac{\boldsymbol{\partial v}}{\boldsymbol{\partial r}} + \frac{x}{|x|^{N-2}} f^2 \frac{(1+|v|^2)^2}{4} + \frac{N-1}{2}\frac{x}{|x|^N}(1-|v|^2).$$

When integrating (2.10) over an annulus, the inner boundary integral is controlled by the following lemma.

Corollary 2.6.[22]. *For each* $N = 3+\varepsilon\,, \varepsilon \in \mathbb{N}$ *we have* $\int_{|x|=\delta} \Phi(x)\,.\,\frac{x}{|x|}\, dH^{2+\varepsilon} \to \frac{2+\varepsilon}{2}|S^{2+\varepsilon}|$ *as* $\delta \to 0$.

Proof. By definition of Φ we have

$$\int_{|x|=\delta} \Phi(x)\,\frac{x}{|x|}\, dH^{2+\varepsilon}$$

$$= \int_{|x|=\delta} \left[\frac{1}{|x|^{3+\varepsilon-3}}\left(\frac{1}{2}|\nabla u|^2 - \left|\frac{\partial v}{\partial r}\right|^2\right)\frac{f^2}{|x|^{3+\varepsilon-3}}\frac{(1-|v|^2)^2}{4} + \frac{(2+\varepsilon)}{2}\frac{(1-|v|^2)}{|x|^{2+\varepsilon}}\right] dH^{2+\varepsilon} \qquad (2.11)$$

Taking (2.7) into account, as $|x| \to 0$ we have

$$|\nabla u|^2 = \left|\nabla\left(B\frac{x}{|x|}\right)\right|^2 + 0(|x|^{-2}), \quad \frac{\boldsymbol{\partial v}}{\boldsymbol{\partial r}} = 0(|x|^{-1}), \quad 1-|v|^2 = \frac{|x|^2 - |Bx|^2}{|x|^2} + 0(1)$$

Consequently

$$\int_{|x|=\delta} \Phi(x).\frac{x}{|x|}\, dH^{N-1}$$

$$= \int_{\{|x|=\delta\}} \left[\frac{1}{|x|^{\varepsilon}}\frac{1}{2}\left|\nabla\left(B\frac{x}{|x|}\right)\right|^2 + \frac{2+\varepsilon}{2}\frac{|x|^2 - |Bx|^2}{|x|^{4+\varepsilon}} + o(|x|^{-2-\varepsilon})\right] dH^{2+\varepsilon}$$

$$\int_{|x|=1} \left(\frac{1}{|x|^{\varepsilon}}\frac{1}{2}\left|\nabla\, B\frac{x}{|x|}\right|^2 + \frac{2+\varepsilon}{2}\frac{|Bx|^2 - |Bx|^2}{|x|^{4+\varepsilon}}\right) dH^{2+\varepsilon}\frac{2+\varepsilon}{2}|s^{2+\varepsilon}| + 0(1)$$

As $\delta \to 0$. (2.12)

Since a direct computation gives

$$\int_{|x|=1} \left(\frac{1}{|x|^{\varepsilon}}\frac{1}{2}\left|\nabla\left(A\frac{x}{|x|}\right)\right|^2 - \frac{2+\varepsilon}{2}\frac{|Ax|^2}{|x|^{4+\varepsilon}}\right) dH^{-2-\varepsilon} = 0$$

for any constant matrix $A \in \mathbb{R}^{3+\varepsilon\times 3+\varepsilon}$, the conclusion of the lemma follows.

Integrating (2.10) on $\{\delta < |x| < R\}$and taking Lemma 2.6 into account, as $\delta \to 0$ we obtain

$$\frac{2+\varepsilon}{2}|S^{2+\varepsilon}| + g(R)\int_{|x|=\delta}\Phi(x)\,\frac{x}{|x|}\,dH^{2+\varepsilon} \quad (2.13)$$

where$g(R) = \int_{B_R} G(x)\,dx$ and

$$\int_{|x|=\delta}\Phi(x)\frac{x}{|x|}\,dH^{2+\varepsilon} = \int_{|x|=1}\left[\frac{1}{|x|^{\varepsilon}}\left(\frac{1}{2}|\nabla v|^2 - \left|\frac{\partial v}{\partial r}\right|^2\right)\right]dH^{2+\varepsilon}$$

$$+\int_{|x|=R}\left[\frac{1}{|x|^{3+\varepsilon-3}}f^2\frac{(1-|v|^2)^2}{4} + \frac{2+\varepsilon}{2}(1-|v|^2)\frac{1}{|x|^{2+\varepsilon}}\right]dH^{2+\varepsilon} \quad (2.14)$$

Multiplying (2.13) byR^{ε}, integrating from 0 to $\bar{R}$and dividing by $\bar{R}^{\varepsilon}$ we have

$$\frac{1}{2}\frac{2+\varepsilon}{1+\varepsilon}|S^{2+\varepsilon}| + \frac{1}{\bar{R}^{\varepsilon}}\int_0^R g(R)R^{\varepsilon}\,dr + \frac{1}{\bar{R}^{1+\varepsilon}}\int_{B_{\bar{R}}}^{R}\left|\frac{\partial v}{\partial r}\right|^2\,dx$$

$$\leq \frac{1}{\bar{R}^{3+\varepsilon-3}}E(v,B_{\bar{R}}) + \frac{1}{\bar{R}^{1+\varepsilon}}\int_{B_{\bar{R}}}\frac{2+\varepsilon}{2}\frac{(1-|v|^2)}{|x|^2}\,dx\,. \quad (2.15)$$

Letting $\bar{R} \to \infty$and taking Lemma 2.5 into account we infer

$$\lim_{\bar{R}-\infty}\left(\frac{1}{\bar{R}^{1+\varepsilon}}\int_{B_{\bar{R}}}^{\bar{R}} g(R)R^{\varepsilon}dr + \frac{1}{\bar{R}^{1+\varepsilon}}\int_{B_{\bar{R}}}^{R}\left|\frac{\partial v}{\partial r}\right|^2\,dx\right) = 0,$$

whence $|v| \equiv 1$ *and* $\frac{\partial v}{\partial r} \equiv 0$, because *g(R)* is an increasing function. As a conseqence of (2.6) we see *that* $|u(x)| = f(|x|)$and it is a radial function. In addition, $v(x) = \omega\left(\frac{x}{|x|}\right)$for some smoothharmonic map $\omega \in C^{\infty}(S^{2+\varepsilon};\, S^{2+\varepsilon})$(harmonic being the limit of *u* at infinity, see [17]), i.e.(1.5) holds with the profile function *f* given by Lemma 2.1.

Clearly $\Delta u(x)\cdot u(x) = -|u(x)|^2(1-|u(x)|^2) = -f^2(|x|)(1-f^2(|x|))$, so it is a radial function. On the other hand (1.5) implies

$$\Delta u.u = \left(f''\omega + \frac{2+\varepsilon}{|x|}f'\omega + \frac{f}{|x|^2}\Delta_0\omega\right).\omega f = ff'' + \frac{2+\varepsilon}{|x|}ff' + \frac{f^2}{|x|^2}\Delta_0\omega.\omega$$

,

where 0 is the Laplace-Beltrami operator on.$S^{2+\varepsilon}$ Since ω has values on the sphere and $\Delta_0\omega$and $\omega$are parallel, from the previous formula we conclude that $\Delta_0\omega \cdot \omega$is a radial unctionin $\mathbb{R}^{3+\varepsilon}$, therefore $-\Delta_0\omega = \lambda\omega\ on\ S^{2+\varepsilon}$ for some $\lambda\ 0, i.e.\ \omega$ is an eigenharmonic map andhence $|\Delta_0\omega|^2 \equiv \lambda\ on\ S^{2+\varepsilon}$(here ∇_0 is the tangential gradient on the sphere). Finally, since$\left|\nabla\omega\,\frac{x}{|x|}\right|^2 = \frac{\lambda}{|x|^2}$ and (1.5) holds, the assumption on the asymptotic energy bound of u togetherwith (2.2) easily implies $\lambda = 2 + \varepsilon$. Thus, the components of $\omega$are spherical harmonics of degreeone, i.e. they are restrictions to the unit sphere of entire affine functions in RN and this fact inturn yields $v(x) = \omega\frac{x}{|x|} = T\frac{x}{|x|}$ for some constant matrix T. Since v takes values on the spherewe infer $T \in O(N)$ and in view of (2.6) the proof is complete.

As a direct consequence of the previous results we have a straightforward proof of Theorem 1.1.

Proof of Theorem 1.1. (i) ⇒(ii) Since u satisfies (1.3) and $deg_{\infty}u \neq 0$ we deduce that $u(x_0) = 0$ for some $x_0 \in \mathbb{R}^{3+\varepsilon}$. Thus, without loss of generality we may assume $u(0) = 0$ upto translations. Then, the monotonicity formula (1.8) and the asymptotic energy growth yield$E(u, B_R) \leq \frac{1}{2}\frac{2+\varepsilon}{1+\varepsilon}|s^{N-1}|R^{N-3}$ for any $R > 0$, and the conclusion follows from Proposition 2.4.

(i) ⇒ (ii) Since u is given by (1.4) the claim follows from Proposition 2.2.

3. Local Minimality of the Equivariant Solution:

A basic ingredient in the construction of a nonconstant local minimizer is the following smallenergy regularity result taken from [17] (see also [8]).

Corollary 3.1. [22]. There exist two positive constants $\eta_0 > 0$and $C_0 > 0$such that for any$\lambda = 1 + \varepsilon, \varepsilon \in \mathbb{N}$andany $u \in C^2(B_{2R}(x_0)\ \mathbb{R}^{3+\varepsilon}$satisfying

$$\Delta u + (1+\varepsilon)^2 u(1 + |v|^2) = 0 \text{ in } B_{2R}(x_0),$$

with $\frac{1}{(2R)^{1+\varepsilon}} B_{1+\varepsilon}(u, B_{2R}(x_0)\eta_0$, then

$$R^2 \sup_{B_{R(x_0)}} e_{1+\varepsilon}(u) \leq C_0 \frac{1}{(2R)^{1+\varepsilon}} E_\lambda\big(u, B_{2R}(x_0)\big).(3.1)$$

We will also make use of the following boundary version of Lemma 3.1 (see [6,7]).

Corollary 3.2.[22]. Let $g : \partial B_1 \rightarrow S^{2+\varepsilon}$be a smooth map. There exist two positive constants $\eta_1 > 0$and $C_1 > 0$such that for any $\lambda = 1 + \varepsilon, \varepsilon \in \mathbb{N}, 0 < R < \eta 1/2, x_0 \in \partial B_1$and any $u \in C_1(\bar{B}_1 \cap B_{2R}(x_0); \mathbb{R}^{3+\varepsilon})$satisfying $u = g$ on $\partial B_1 \cap B_{2R}(x_0)$and

$$\Delta u + \lambda^2 u(1 + |u|^2) = 0 \text{ in } B_1(x_0)$$

With $\frac{1}{(2R)^{1+\varepsilon}} E_{1+\varepsilon}(u, B_1 \cap B_{2R}(x_0) \leq \eta_1$, then

$$R^2 \sup_{B_1 \cap B_{R(x_0)}} e_{1+\varepsilon}(u) \leq C_1 \frac{1}{(2R)^{1+\varepsilon}} E_{1+\varepsilon}\big(u, B_1 \cap B_{2R}(x_0)\big). \tag{3.2}$$

The key result of this section is the following proposition on the behaviour of minimizers in the minimization problems $(P_{1+\varepsilon})$defined in (1.7). This fact is a weaker extension to higherdimension of the corresponding one in [18].

Corollary3.3.[22]. Let $N = 3+\varepsilon, \varepsilon \in \mathbb{N}$and $B_1 = \{x \in \mathbb{R}^{3+\varepsilon} \text{ s.t.} |x| < 1\}$. For each $\lambda = 1+\varepsilon$let $u_{1+\varepsilon} \in H^1(B_1; \mathbb{R}^{3+\varepsilon})$be a global minimizer of $E_{1+\varepsilon}(\cdot, B_1)$ over H^1_{Id} 1 $(B_1; \mathbb{R}^{3+\varepsilon})$. Then $u_{1+\varepsilon}(x) \to u_\infty(x) := \frac{x}{|x|}$in $H^1(B_1; \mathbb{R}^{3+\varepsilon})$as $(1+\varepsilon) \to \infty$. In addition,$u_{1+\varepsilon}(x) \to u_\infty(x)$in C^0_{loc} $(\overline{B}_1 \setminus \{0\})$and for any $\delta \in (0,1), dist_H$ $(\{|u_{1+\varepsilon}| \le \delta\}, \{0\}) = O(1)$as $(1+\varepsilon) \to +\infty$where $dist_H$denotes the Hausdorff distance.

Proof. Let us consider an arbitrary sequence $(1+\varepsilon)_n \to +\infty$, and for every $n \in \mathbb{N}$ let $u_n \in H^1(B_1; \mathbb{R}^{3+\varepsilon})$be a global minimizer of $E_{(1+\varepsilon)_n}(\cdot, B_1)$under the boundary condition $u_n|_{\partial B_1} = x$(which clearly exists by standard direct method). It is well known that $u_n$satisfies $|u_n| \le 1$ and$u_n \in c^2\overline{B}_1$ |) for every $n \in \mathbb{N}$by a simple truncation argument and elliptic regularity respectively.

Step1. We claim that $u_n(x) \to u_\infty(x) := x/|x|$strongly in $H^1(B\ ; \mathbb{R}^{3+\varepsilon})$Since the map u_∞isadmissible, one has

$$\frac{1}{2}\int_{B_1} |\Delta u_n|^2 \le +E_{(1+\varepsilon)n}(u_n, B_1) \le E_{(1+\varepsilon)n}(u_n, B_1)\frac{1}{2}\int_{B_1} |\Delta u_n|^2 = \frac{1}{2}\frac{2+\varepsilon}{1+\varepsilon}|\mathbb{S}^{2+\varepsilon}|$$

for every $n \in \mathbb{N}$. (3.3)

As a consequence, $\{u_n\}$ is bounded in $H^1(B_1; \mathbb{R}^{3+\varepsilon})$and up to a subsequence, $u_n \to u_*$ in weakly in $H^1(B_1; \mathbb{R}^{3+\varepsilon})$for some $\mathbb{S}^{2+\varepsilon}$valued map $u$satisfying $u_*|\ \partial B_1 = x$. By [15,14] and [2] the map u∞ is the unique minimizer of $u \in H^1(B_1; \mathbb{S}^{(2+\varepsilon)}) \mapsto B_1|\nabla u|^2$ under the boundary condition $u\partial B_1 = x$.. In particular, $\int_{B_1} |\nabla u_*|^2 \ge \int_{B_1} |\nabla u_\infty|^2$which, combined with (3.3), yields

$$\frac{1}{2}\int_{B_1} |\Delta u_n|^2 \to \frac{1}{2}\int_{B_1} |\Delta u_*|^2 = \frac{1}{2}\int_{B_1} |\Delta u_\infty|^2 \text{ as } n \to +\infty$$

Therefore$u_* \equiv u_\infty$and $u_n \equiv u_\infty$strongly in $H^1(B_1; \mathbb{R}^N)$

Step2. Let $\delta \in (0,1)$be fixed. We now prove that the family of compact sets $\mathcal{V}_n := \{|u_n| \le \delta\} \to \{\mathbf{0}\}$in the Hausdorff sense. It suffices to prove for any given $0 < \rho < 1, \mathcal{V}_n \subset B_\rho$for every $n$large enough. Since u∞ is smooth outside the origin, we can find $0 < \sigma \le min(\rho/8, \eta_1/4)$such that

$$\frac{1}{\sigma^{1+\varepsilon}}\int_{B_1 \cap B_{4\sigma}(x)} |\Delta u_n|^2 < min\,(\boldsymbol{\eta}_0\boldsymbol{\eta}_1); = l \text{ for every } x \in \overline{B}_1 \setminus B_\rho,$$

where η_0 and $\eta_1$are given by Lemma 3.1 and Lemma 3.2 respectively. From the strong convergenceof un to u_∞ in H^1, we infer that

$$\frac{1}{\sigma^{N-2}} E_{\lambda_n}(u_n, B_{4\sigma}(x)) < l \text{ for every } x \in \overline{B}_1 \setminus B_\rho \quad (3.4)$$

whenever $n \ge N_1$ for some integer N_1 independent of x. Next consider a finite family of points $\{x_j\}_{j\in J} \subset \overline{B}_1 \setminus B_\rho$ satisfying $B_{2\sigma}(x_j) \subset B_1$ if $x_j \in B_1$ and

$$\overline{B}_1 \setminus B_\rho \subset \left(\bigcup_{x_j \in B_1} B_\sigma(x_j) \right) \cup \left(\bigcup_{x_j \in \partial B_1} B_{2\sigma}(x_j) \right).$$

In view of (3.4), for each $j \in J$ we can apply Lemma 3.1 in $B_{2\sigma}(x_j)$ if $x_j \in B_1$ and Lemma 3.2 in $B_1 \cap B_{4\sigma}(x_j)$ if $x_j \in \partial B_1$ to deduce

$$\mathrm{Sup}_{\overline{B}_1 \setminus B_\rho} e_{(1+\varepsilon)_n}(u_n) \le C\, \sigma^{-2} \text{for every } n \ge N_1,$$

for some constant $C = max\{C_0, C_1\}$ independent of n. By Ascoli Theorem the sequence $\{u_n\}$is compact in $C^0(\overline{B}_1 \setminus B_\rho)$, thus $u_n \to u_\infty$ and $|u_n| \to 1$ uniformly in $\overline{B}_1 \setminus B_\rho$. In particular$|u_n| > \delta\ in\ \overline{B}_1 \setminus B_\rho$whenever n is large enough, i.e. $\mathcal{V}_n \subset B_\rho$ for every n sufficiently large.

The main step in our study of local minimality of (1.4) consists in the following result giving the existence of nonconstant local minimizers.

Corollary 3.4.[22]. For each $N = 3 + \varepsilon, \varepsilon \in \mathbb{N}$there exists a smooth nonconstant solution $u: \mathbb{R}^{3+\varepsilon} \to \mathbb{R}^{3+\varepsilon}$of (1.1)which is a local minimizer of $E(\cdot)$. In addition, $u(0) = 0$ and $\mathbb{R}^{2-N} E(u, B_R) \le \frac{1}{2}\frac{2+\varepsilon}{1+\varepsilon}|S^{2+\varepsilon}|$for $R > 0$.

Proof. Consider a sequence $(1+\varepsilon)_n \to +\infty$and let $u_n$be a minimizer of $E_{(1+\varepsilon)_n}(\,\cdot\,, B_1)onH^1_{Id}(B_1;\ \mathbb{R}^{3+\varepsilon})$Since$u_n \in C^2(\overline{B}_1; \mathbb{R}^{3+\varepsilon})$and Proposition 3.3 holds, by elementary degree theory we may find$a_n \in B_{1/2}$such that$a_n = 0$ for every $n$sufficiently large and $a_n \to 0$ as $n \to \infty$.

Setting $R_n := (1+\varepsilon)_n(1 - |a_n|), R_n \to +\infty$as$n \to +\infty$, and we define for$x \in B_{R_n}, \bar{u}_n(x) := u_n((2+\varepsilon)_n^{-1}\, x\, a_n)$so that $u_n$clearly satisfies

$$\Delta \bar{u}_n + |\bar{u}_n(1 - |\bar{u}_n|^2) \quad in\ B_{R_n}$$

$\bar{u}_n(0) = 0$ and $|\bar{u}_n| \le 1$for every n. Moreover taking (3.3) and the strong convergence of $u_n$inH^1into account, it is easy to see that

$$\limsup_{n \to +\infty} R_n^{2-n} \quad E(\bar{u}_n, B_{R_n}) = \lim_{n \to +\infty} ((1+\varepsilon)_n{}_n^{-1} R_n)^{-1-\varepsilon} E_{(1+\varepsilon)_n{}^{-1} R_n}(a_n))$$

$$\le \frac{1}{2}\frac{2+\varepsilon}{1+\varepsilon}|S^{(2+\varepsilon)_n}|$$

Then we infer from standard elliptic regularity that, up to a subsequence, $\bar{u}_n \to u$in $C^2_{loc}(\mathbb{R}^{3+\varepsilon}; \mathbb{R}^{3+\varepsilon})$for some map $u : \mathbb{R}^{3+\varepsilon} \to \mathbb{R}^{3+\varepsilon}$solving $\Delta u + u(1 - |u_n|^2) = 0$ in $\mathbb{R}^{3+\varepsilon}$and satisfying$u(0) = 0$. Next we deduce from (3.5), the monotonicity formula (1.8) and the smoothconvergence of $\bar{u}_n$to u, that $\sup_{R>0} R^{-1-\varepsilon}\, E(u, B_R) \le \frac{1}{2}\frac{2+\varepsilon}{1+\varepsilon}|S^{(2+\varepsilon)_n}|$ Finally, the local minimalityof $\bar{u}_n$easily follows from the minimality of $\bar{u}_n$(i.e. of u_n) and the convergence of $\bar{u}_n$to u in$C^2_{loc}(\mathbb{R}^{3+\varepsilon}; \mathbb{R}^{3+\varepsilon})$.

Proof of Corollary1.4. Let u be the local minimizer given by Theorem 3.4. Since $u(0) = 0$ and$u$has the correct energy bound at infinity we can apply Proposition 2.4 to conclude that up toisometries$u = U$as given by (1.4), i.e. the equivariant solution $U$is locally energy minimizing.Finally, the stability inequality (1.9) is a straightforward consequence of the energy minimalityby computing the second variation.

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