The Geometrical Formulation of General Symmetric Dynamical System

الصياغة الهندسية للنظم الديناميكية العامة المتماثلة

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المستخلص

من المعروف أن النظام الديناميكي يمكن نمزجته كمتعدد طيات سمبلكتيكي (M,ω) مذود بصياغة سمبلكتيكية ω باستخدام هذه الحقيقة بينا أن أي نظام ديناميكي يمكن تمثيله هندسيا كمدار لزمرة لي التماثلية علي جبر لي الثنائي لهذه الزمرة.

Abstract

It is well known that a dynamical system may be modeled as a symplectic manifold (M,ω) with a symplectic geometric form ω. Using this fact we showed that any dynamical system can be represented geometrically as an orbit of the symmetry Lie group on the dual of the Lie Algebra of this group.

Keywords: Dynamical System, Lie Group, adjoint Representation, Dual Representation.

1. Introduction:

In mechanics, symmetry can be used to reduce the dynamic, that is, to transform the equations of motion into a few set of equations. There are several general kinds of reduction, all based on Lie group actions, and all with the property that the reduced system inherits the mathematical structure (**Lagrangian or Hamiltonian**) of the original system.

Historically Hamiltonian reduced the two second order differential equations of mechanics into a set of two first order differential equations. This inspired mathematician to formulate dynamics in the reduction skeme using the notation of symmetry group. In fact the set of two first order differential equations of Hamiltonian can be moreover reduced into only one equation using a differential form called a symplectic form that gives as an invariant structure on the phase space.

The underlying geometric structure can best be utilized in the construction of general symmetric dynamical system characterized by a symmetry group G . In fact this paper showed that any dynamical system is interpreted as an orbit of the symmetry Lie group on the dual of the Lie Algebra of this group.

2. Dynamical System (Description via Lagrangian System and Hamiltonian System):.

2.1. Introduction

We have been study classical mechanics as formulated by Sir Isaac Newton, this is called Newtonian mechanics is mathematically fairly straightforward, and can be applied to a wide variety of problem. It is not unique formulation of mechanics, however, other formulation are possible. Here we will look at two common alternative formulations of classical mechanics: Lagrangian mechanics and Hamiltonian mechanics.

It is important to understand that all of these formulations of mechanics equivalent. In principle, any of them could be used to solve any problem in classical mechanics. The reason they are important is that in some problems one of the alternative formulations mechanics may lead to equations that are much easier to solve than the equations that arise from Newtonian mechanics.

Unlike Newtonian mechanics, neither Lagrangian nor Hamiltonian mechanics requires the concept of force, instead, these systems are expressed in terms of energy.

Although we will be looking at the equations of mechanics in one dimensions, all these formulations of mechanics may be generalized to two or three dimensions.

2.2. *Lagrangian Mechanics*

The first alternative Newtonian mechanics we will look at is Lagrangian mechanics. Using Lagrangian mechanics instead of Newtonian mechanics is sometimes advantageous in certain problem, where the equations of Newtonian mechanics would be quite difficult to solve.

In Lagrangian mechanics, we begin by defining a quantity called the Lagrangian (L), which is defined as the difference between the kinetic energy K and potential energy U :

$$L = K - U$$

Since the kinetic is a function of velocity v and potential energy will typically be a function of position x , the Lagrangian will (in one dimension) be a function of both x and v : $L(x;v)$. The motion of a particle is then found by solving Lagrange's equation; in one dimension it is

$$\frac{d}{dt}\left[\frac{\partial L}{\partial V}\right]-\frac{\partial L}{\partial V}=0 \qquad (1)$$

2.3. Hamiltonian Mechanics

The second formulation we will look at is Hamiltonian mechanics. In this system, in place of the Lagrangian we define a quantity called the Hamiltonian, to which Hamilton's equations of motion are applied. While Lagrange's equation describe the motion of a particle as a single second – order differential equation, Hamilton's equations describes the motion as a coupled system of two first – order differential equations.

One of the advantages, of Hamiltonian mechanics is that it is similar in form to quantum mechanics, the theory that describes the motion of particles at very tiny (subatomic) distance scales. An understanding of Hamiltonian mechanics provides a good introduction to the mathematics of quantum mechanics.

The Hamiltonian H is defined to be the sum of the kinetic and potential energies:

$$H \equiv K + U$$

Here the Hamiltonian should be expressed as a function of position x and momentum p rather than x and v, as in the Lagrangian), so that $H = H(x, p)$. This means that the kinetic energy should be written as $K = p^2/2m$, rather than $K = mv^2/2$. Hamilton's equations in one dimension have the elegant nearly symmetrical form

$$\frac{dx}{dt}=\frac{\partial H}{\partial p} \qquad (3)$$

$$\frac{dp}{dt}=-\frac{\partial H}{\partial x} \qquad (4)$$

3. Symplectic Description:

We show that a symplectic description provided by a symplectic form ω is in fact equivalence to Hamilton equations given by (3) and (4). We shall use the local form of ω which is

$$\omega = dq \wedge dp$$

And the coordinate description of a Hamiltonian vector field

$$X = \sum_{i=1}^{n} \dot{q}_i \frac{\partial}{\partial q_i} + \dot{p}_i \frac{\partial}{\partial p_i}$$

Then

$$X \lrcorner \omega = \left(\sum_{i=1}^{n} \dot{q}_i \frac{\partial}{\partial q_i} + \dot{p}_i \frac{\partial}{\partial p_i} \right) \lrcorner \omega$$

$$= \dot{q}_i \, dp - \dot{p}_i \, dq$$

Using Hamilton's equations (3) and (4) we get

$$X \lrcorner \omega = \frac{\partial H}{\partial p} dp + \frac{\partial H}{\partial q} dq$$

$$= dH$$

4. Symmetry Lie Group

A group G is called a Lie group if it is a smooth manifold whose group operations (product and inversion) are differential.

Let ψ be a $1-$ parameter group action on a differentiable manifold M, $\psi : G \times M \to M$, $\psi(\varepsilon, x)$ then the infinitesimal generator of this action is given by

$$V_x = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \psi(\varepsilon, x)$$

Similarly one get the flow or the one – parameter group generated by a vector field V as

$$\exp(\varepsilon V)x = \psi(\varepsilon, x)$$

Let a map $L_{g_1} : G \to G$ be defined by $L_{g_1} g = g_1 g$. This actions is called Left action. Then we have

$$(L_{g_1})_* : T_e G \to T_e G$$

A vector field V is called Left invariant if $(L_{g_1})_* V_g = V_{g_1 g}$, shortly $L_* V = V$. Similarly a differential form ω is called Left invariant $(L_{g_1})_* \omega_g = \omega_{g_1 g}$, shortly $L^* \omega = \omega$.

The set of all left invariant vector field on M is called a Lie Algebra of G and denoted by Q.

Geometrically is the tangent space $T_e G$ at the identity e.

5. Representation Space (adjoint representation $\Rightarrow Q^*$)

Let G be a connected simply connected Lie group and let Q be the Lie algebra of G : Q can be identified with the tangent space to G at the identity, $e \in G$.

Each $g \in G$ defines a diffeomorphism τ_g of G which preserves the identity:

$$\tau_g : \mapsto gxg^{-1}; \qquad x \in G$$

The derivative of τ_g at e is therefore a linear transformation of Q, denoted Ad_g, the map $g : \mapsto Ad_g$ is called the adjoint representation of G; it satisfies $Ad_{gg'} = Ad_g Ad_{g'}$ for all $g, g' \in G$.

Thus G acts on Q as a group of linear transformations. This action induces a second action Ad' on the dual space Q^*, called the coadjoint representation.

Explicitly:

$$\left(Ad'_g f\right)(X) = f\left(Ad_{g^{-1}} X\right) \; f \in Q^*, X \in Q, \; g \in G. \qquad (5)$$

For simplicity, $.Ad'_g \cdot f$ will be written $g \cdot f$.

Suppose now that

$$M_{f_\circ} = \{g \cdot f_\circ | g \in G\}, \quad f_\circ \in Q^* \qquad (6)$$

Is an orbit in Q^* and that $f \in M_{f_\circ}$. Each $X \in G$ generates a one parameter subgroup of G and hence defines a flow on $M_{f_\circ}$; let ζ_X be the tangent vector field flow. The map:

$$Q \to T_f M_{f_\circ} : X \mapsto X_f = (\zeta_X)_f$$

Is linear and surjective (since the action of G on $M_{f_\circ}$ is transitive). Also, if $X, X' \in Q$, then $X_f = X'_f$ if, and also if, where $Z \in Q$ is such that $\exp(tZ)$ leaves f invariant for each $t \in R$.

From the form of the coadjoint representation, this is equivalent to:

$$f([Z, X]) = 0 \qquad \forall X \in Q \qquad (7)$$

Thus the quantity ω_f, given by:

$$\omega_f(X_f, Y_f) = f([X, Y]); \qquad X, Y \in Q \qquad (8)$$

Is a well-defined skew symmetric bilinear form on $T_f M_{f_\circ}$; it is also non-degenerate since, if $X \in Q$ then:

$$\omega_f(X_f, Y_f) = 0 \qquad \forall Y \in Q \qquad (9)$$

If, and only if, each $\exp(tX)$ leaves f invariant, that is, if and only if, $X_f = 0$.

As f varies, ω_f defines a non-degenerate 2-form $\omega \in \Omega^2(M_{f_\circ})$; to show that ω is, in fact, a symplectic structure, it is only necessary to prove that $d\omega = 0$.

Not surprisingly, this follows from the Jacobi identity in Q . In fact, if $X, Y \in Q$ then:

$$[\zeta_X, \zeta_Y]_f = [X, Y]_f \qquad (10)$$

Thus if $X, Y, Z \in Q$ then:

$$(d\omega(\zeta_X, \zeta_Y, \zeta_Z))_f = \sum_{cyclic} X_f(\omega(\zeta_Y, \zeta_Z)) - f([[X, Y], Z]) \qquad (11)$$

The second term is zero by the Jacobiidentity in Q . The first term can be computed using the fact that, for fixed $X \in Q$, the rate of change of $f(X)$ along ζ_Z, where $Z \in Q$, is $f([Z, X])$ since , for small $t \in R$:

$$((\exp(-tZ)) \cdot f)(X) = f(\exp(tZ) \cdot X) = f(X + t[Z, X]) + O(t^2) \qquad (12)$$

Thus :

$$X_f(\omega(\zeta_Y, \zeta_Z)) = f([X, [Y, Z]]) \qquad (13)$$

And so the first term also vanish by the Jacobi identity.

Finally ω is invariant under the action of G on $M_{f_\circ}$ since , for any $X, Y, Z \in Q$:

$$\begin{aligned} &((\xi_{\zeta_X}\omega)(\zeta_X, \zeta_Z))_f \\ &= X_f(\omega(\zeta_Y, \zeta_Z)) - \omega([\zeta_X, \zeta_Y], \zeta_Z)_f - \omega([\zeta_Y, [\zeta_X, \zeta_Z]])_f \\ &= f([X, [Y, Z]]) + ([Z, [X, Y]] + [Y, [Z, X]]) \\ &= 0 \end{aligned}$$

6. Construction of the dynamical system using the dual representation

From the above information each orbit in Q^* has the structure of an classical phase space on which G acts as a transitive invariance group.

The importance of this result is that essentially all classical phase spaces which admit G as a transitive invariance group (that is all homogeneous symplectic G manifolds) arise in this way.

First , some notation . A symplectic manifold (M, ω) is called a Hamiltonian G - space for a Lie group G if there is given a Lie algebra homomorphism :

$$\lambda : G \rightarrow \overset{\infty}{\underset{R}{C}}(M) \; : \; X \in \phi_X$$

From the Lie algebra of G in to the space of real functions (observables) on M such that Conversely, each orbit $M_{f_\circ}$ is a Hamiltonian G - space : λ is defined by :

$$(\lambda((X))(f) = f(X); f \in M_{f_\circ}, X \in Q) \qquad (14)$$

1. Each Hamiltonian vector field $\zeta_X = \zeta_{\phi_X}$ is complete.

2. Any two points $m_1, m_2 \in M$ can be joined by an integral curve of ζ_X for some $X \in G$.

Every Hamiltonian G - space is also a homogeneous symplectic G - manifold (14): the action of each $g \in G$ is defined by integrating the (complete) Hamiltonian vector field of the corresponding generator in G. Moreover, and this is the important point, every Hamiltonian G - space is a covering space of an orbit in (14) G^*. The proof of this is almost trivial : if (M, ω) is a Hamiltonian G - space then the map

$$M \to G^* \ : \ m \mapsto f_m$$

Defined by:

$$f_m(X) = \phi_X(m); \quad m \in M, \quad X \in G \qquad (15)$$

Commutes with the actions of G on M and G^* and so maps M onto an orbit in G^*: it is not hard to see that it is, in fact, a covering map. Suppose now that there is given a classical system with a phase space (M, ω) and a transitive invariance group G. If it possible to find a map $\lambda : G \to \underset{R}{\overset{\infty}{C}}(M)$ which generates the action of G and which makes (M, ω) into a Hamiltonian G-space, then (M, ω) can be identified with a covering space of an orbit in G^* (in fact; if M is simply connected then it must actually be an orbit in G^*), (M, ω) can then be classified purely in terms of the structure of G.

For λ to exist, two conditions must be satisfied (it is these that are embodied in the qualification 'essentially'): First, each generator $X \in G$ of G defines a one parameter group of canonical transformations of M, and hence a locally Hamiltonian vector field ζ_X : for λ to exist, each ζ_X must in fact be globally Hamiltonian. This will be so if M is simply connected or (as in the case of $SO(3)$) if $G = [G, G]$ for example; if G is semi – simple): Secondly, even if ϕ_X can be found for each $X \in G$ individually, it will not necessarily follow that λ preserves, that is that:

$$\phi_{[X,Y]} = [\phi_X, \phi_Y] \qquad X, Y \in G. \qquad (16)$$

The condition that each ϕ_X can be chosen so that this is true involves the homology of G. However, it is always true (provided the first condition is satisfied) that λ can be found for some central extension of G.

At the purely classical level, therefore, this construction provides an elegant classification scheme for the elementary systems with a given invariance group. At the quantum level it assumes a more important role. For suppose that (M, ω) is a quantizable symplectic manifold and that G should act as a symmetry group on the phase space of the underlying quantum system, also, according to the argument given in S_5, this action should be irreducible . If (M, ω) is, in fact, a Hamiltonian G - space, and G is simply connected (if G is not simply connected then this gives a representation of the universal covering group . For example if $G = SO(1,3)$ then the construction gives representation of $SL(2, C)$ and thus leads naturally to the spinor concept), then the first part, at least, will automatically be achieved by geometric quantization: each generator $X \in G$ is associated with a classical observable ϕ_X and hence with a vector field $n_X = n_{\phi_X}$ on the pre quantization line bundle, L. This vector field will be complete (since ζ_X is complete) and will generate a one parameter family of unitary transformations of $\Gamma(L)$. Thus $\exp(X)$ and hence G

has a natural action on $\Gamma(L)$. An irreducible action can usually be achieved by choosing a G -invariant polarization of M .

Conclusion:

The study of dynamical systems has attracted the attention of several mathematics researcher. This is mainly because of the importance of this study to other branches of applied Sciences and Engineering. For instance, Control Theory essentially involves dynamical system. In our paper we have provided the general geometrical formulation of dynamical systems, using abstract spaces and transformation Lie groups to investigate the existence of solutions.

Our main objective has been to construct a geometrical model via which one can study the structure of dynamical system. This structure facilitate the existence and classification of solutions of geometrical set up.

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