# Symmetry and Invariants


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**المستخلص**

في هذه الورقة تم توضيح أن هناك علاقة بين جبر لي لزمر تماثل نظم لاجرانج والكميات المحافظة . من خلال استخدام نظرية نوثر لتحقيق هذا الهدف. كذلكتم استعمال هذه العلاقة لإيجاد ثوابت النظم الميكانيكية.

## Abstract

In this paper we showed that there is a relationship between the Lie algebra of the symmetry groups for a Lagrangian system and the conserved quantities. We utilized Noether theorem to achieve this goal. We used this relationship to obtained the invariants of mechanical systems.


**Keywords: Dynamical System, Lie Group, Lorentz Theorem , Noether Theorem, Invariants, symmetries.**

**1. Introduction:**

Mechanics has two main points of view , (Lagrangian mechanics and Hamiltonian mechanics)In one sense, Lagrangian mechanics is more fundamental, since it is based on variational principles and it is what generalizes most directly to the general relativistic context. In another sense, Hamiltonian mechanics is more fundamental, since it is based directly on the energy concept and it is what is more closely tied to quantum mechanics .

Symmetry plays an important role in mechanics, from fundamental formulations of basic principles to concrete applications, such as stability criteria for rotating structures.

Lie symmetries is a powerful method for the determination of solutions in the theory of differential equations. A Lie symmetry is important as it provides invariants which can be used to write a new differential equation with less degree of freedom.

The Noether symmetry is an invariance of the Lagrangian action under the infinitesimal transformations of time and coordinates. Besides the Noether symmetry, there are two other important symmetries, i.e. the Lie symmetry and the form invariance. The Lie symmetry is a kind of invariance of the differential equations under the infinitesimal transformations of time and coordinates. The form invariance is a kind of invariance under which the transformed dynamical functions still satisfy the original differential equations of motion. A Noether symmetry can lead to a conserved quantity according to the Noether theory. A Lie symmetry or a form invariance can also lead to a conserved quantity under certain conditions. The conserved quantities deduced directly by the Noether symmetry, the Lie symmetry and the form invariance are called the Noether conserved quantity.

**2.Lorentz Group and its Representations:**

The Lorentz group starts with a group of four-by-four matrices performing Lorentz transformations on the four-dimensional Minkowski space of . The transformation leavesinvariant the quantity . There are three generators of rotations and threeboost generators. Thus, the Lorentz group is a six-parameter group.

It was Einstein who observed that this Lorentz group is applicable also to the four-dimensional energy and momentum space of . In this way, he was able to derivehis Lorentz-covariant energy-momentum relation commonly known as . This trans-formation leaves invariant. In other words, the particle mass is a Lorentz invariant quantity.

**2.1 Generators of the Lorentz Group**

**Let us start with rotations applicable to the coordinates. The four-by-four matrix for this operation is**

$$Z(\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix} \tag{1}$$

which can be written as

$$Z(\phi) = \exp(-i\,\phi J_3) \tag{2}$$

With

$$J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \tag{3}$$

The matrix $J_3$ is known as the generator of the rotation around the $z$ axis. It is not difficult to write the generators of rotations around the $x$ and $y$ axes, and they can be written as $J_1$ and $J_2$ respectively, with

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \quad , \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4}$$

These three rotation generators satisfy the commutation relations

$$\left[J_i, J_j\right] = i\,\varepsilon_{ijk} J_k \tag{5}$$

The matrix which performs the Lorentz boost along the $z$ direction is

$$B(\eta) = \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6}$$

With $B(\eta) = \exp(-i\,\eta K_3)$, (7)

And the generator

$$K_3 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{8}$$

It is then possible to write the matrices for the generators $K_1$ and $K_2$, as

$$K_1 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \tag{9}$$

$$\left[J_i, K_j\right] = i\varepsilon_{ijk} K_k, \text{and } \left[K_i, K_j\right] = -i\varepsilon_{ijk} J_k \tag{10}$$

There are six generators of the Lorentz group, and they satisfy the three sets of commutation relations given in Eq. $(5)$ and Eq. $(10)$. It is said that the Lie algebra of the Lorentz group consists of these sets of commutation relations.

These commutation relations are invariant under Hermitian conjugation. While the rotation generator is Hermitian, the boost generators are anti-Hermitian

$$\overset{\dagger}{\underset{i}{J}} = J_i \ , \ \text{while} \ \overset{\dagger}{\underset{i}{K}} = -K_i \ . \tag{11}$$

Thus, it is possible to construct two four-by-four representations of the Lorentz group, one with $K_i$ and the other with $-K_i$. For this purpose we shall use the notation (Berestetskii 1982, Kim and Noz 1986)

$$\dot{\underset{i}{K}} = -K_i \tag{12}$$

Since there are two representations, transformations with $K_i$ are called the covariant transformations, while those with $\dot{\underset{i}{K}}$ are called contravariant transformations.

**1. Lie Symmetry Group:**

In group theory , the symmetry group of a geometric object is the group of all transformations under which the object is invariant , endowed with the group operation of composition .

A Lie group is a smooth $(C^\infty)$ manifold $G$ equipped with a group structure so that the map

$M : G \times G \to G \ \ (x, y) \mapsto xy$ and $\ell : G \to G \ \ x \mapsto x^{-1}$ are smooth.

Let $\theta$ be a 1-parameter group action on a differentiable manifold

$M \ , \quad \theta : G \times M \to M \qquad \theta(\varepsilon, x)$ then the infinitesimal generator of this action given by

$$V_x = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \theta(\varepsilon, x)$$

The one-generated group generated by vector field $V$ as

$$\exp(\varepsilon V)x = \theta(\varepsilon, x)$$

Let a map $L_x : G \to G \quad y \mapsto xy$ (where $L_x$ is the translation map). This actions is called Left action. Then we have:

$$(L_x)_* : T_y G \to T_y G$$

to the Lagrangian. But first we will quickly review the main physical implication of symmetry, which is that it implies the existence of conserved charges.

**6.1.2 Noether's theorem (a conserved current ):**

Themainsymmetries we will beinterested in arecontinuous symmetries, where the transformation parameters are real (or complex) numbers, not just integers.

In field theory, the main tool with which to quantity the concept of symmetry is Noether's theorem, loosely speaking, it states that:

To every continuous symmetry of the action there corresponds a conserved current.

Here by a conserved current we mean a quantity, $j^{\mu}(x)$, built up of the fields in our Lagrangian , which satisfies the continuity equation

$$\partial_{\mu} j^{\mu}(x) = 0 \qquad (17)$$

This is a local condition, but it can be shown to imply that there exists a corresponding charge, the integral of the zero-component of the current over a large volume, which is constant in time:

$$Q = \int d^3x\, j^0(x) \Rightarrow \frac{\partial}{\partial t} Q = \int d^3x \frac{\partial j^0(x)}{\partial t} = \int d^3x\, \vec{\nabla} \cdot \vec{j}(x) = 0 \qquad (18)$$

Where we have used the divergence theorem in the last step.

Below I will sketch the derivation of Noether's theorem. Consider a Lagrangian $L(\phi^i)$ depending on a collection of fields $\{\phi^i\}$. Let us assume that $L$ is invariant under a specific combined transformation of those fields $\delta\phi^i$ . ( In other words, the transformation $\delta\phi^i$ is a symmetry). Performing the variation, we find

$$\delta L = \frac{\partial L}{\partial \phi^i} \delta\phi^i + \frac{\partial L}{\partial(\partial_\mu \phi^i)} \delta(\partial_\mu \phi^i) \qquad (19)$$

Now use the identity

$$\partial_\mu \left[ \frac{\partial L}{\partial(\partial_\mu \phi^i)} \delta\phi^i \right] = \partial_\mu \left( \frac{\partial L}{\partial(\partial_\mu \phi^i)} \right) \delta\phi^i + \frac{\partial L}{\partial(\partial_\mu \phi^i)} \delta(\partial_\mu \phi^i) \qquad (20)$$

To rewrite (H5)

$$\delta L = \left[ \frac{\partial L}{\partial \phi^i} - \partial_\mu \left( \frac{\partial L}{\partial(\partial_\mu \phi^i)} \right) \right] \delta\phi^i + \partial_\mu \left[ \frac{\partial L}{\partial(\partial_\mu \phi^i)} \delta\phi^i \right] = 0 \qquad (21)$$

The full expression is zero according to the action principle. But the terms in the square brackets are simply the Euler-Lagrange equations, and are equal to zero on their own. This implies that the second term is zero as well: But this equation has the form of a conserved current:

$$\partial_\mu j^\mu = 0, where\ \partial_\mu j^\mu = \frac{\partial L}{\partial(\partial_\mu \phi^i)} \delta\phi^i \qquad (22)$$

Whatwe have shown (following Emily Noether) is that an invariance of the action ($\delta L = 0$under the variation $\delta\phi^i$ **)** has led to a conserved current ( where $j^\mu(x)$ is the conserved Noether's current) .

**6.1.3 Conserved quantity**

In mathematics , a conserved quantity of a dynamical system is a function $H$ of dependent variables that is constant along each trajectory of the system.

A conserved quantity can be useful tool for qualitative analysis. Since most laws of physics express some kind of conservation , conserved quantities commonly exist in mathematics model of real systems. For example any mechanics model will have energy as a conserved quantity so long as the forces involved are conservative.

**6.1.4 Differential equation**

For a first order system of differential equation

$$\frac{dr}{dt} = f(r,t)$$

Where bold $\frac{dr}{dt}$ indicates vector quantities , $f(r,t)$ is a vector-valued function . $H(r)$ is a conserved quantity of the system if for all time and initial condition in some specific domain

$$\frac{dH}{dt} = 0$$

By using the multivariate chain rule:

$$\frac{dH}{dt} = \nabla H \cdot \frac{dr}{dt} = \nabla H \cdot f(r,t)$$

$$\nabla H \cdot f(r,t) \tag{23}$$

Which contains information specific to the system and can be helpful in finding conserved quantities , or establishing whether or not a conserved quantity exists.

**5. Hamiltonian and Lagrangian Mechanics**

**5.1 Hamiltonian mechanics**

Here we discuss some special nonlinear ODE that have some very nice properties .

**Definition 5.1.1**

Let $E$ be an open subset of $\mathbb{R}^{2n}$ and let $H \in C^{2}(E)$ where $H = H(x,y)$ with $x, y \in \mathbb{R}^{n}$ . The system

$$x' = \frac{\partial H}{\partial y} \quad , \quad y' = -\frac{\partial H}{\partial x}$$

Where

$$\frac{\partial H}{\partial x} = \left(\frac{\partial H}{\partial x_1}, \frac{\partial H}{\partial x_2}, \ldots, \frac{\partial H}{\partial x_n}\right)^T \quad , \quad \frac{\partial H}{\partial y} = \left(\frac{\partial H}{\partial y_1}, \frac{\partial H}{\partial y_2}, \ldots, \frac{\partial H}{\partial y_n}\right)^T \tag{24}$$

Is called a Hamiltonian system with $n$ degrees of freedom on $E$ .

**Properties of Hamiltonian System 5.1.2**

**(i)** The equilibrium points of the Hamiltonian system correspond to the critical points of $H$ .

- An equilibrium points $(a_x\ a_y)$ are called non-degenerate if the determinant second derivative of $H$ evaluated at the equilibrium point is nonzero , i.e. $\left|\frac{\partial^2 (a_x a_y)}{\partial (x,y)^2}\right| \neq 0$

**(ii)** Hamiltonian systems are conservative , i.e., $H(x,y)$ remains constant along the trajectories

**(iii)** For a system defined by the Hamiltonian $H$ a function $f$ of the generalized coordinates $q$ and generalized momentum $\vec{p}$ has time evaluation

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t} \tag{25}$$

And hence is conversed if and only if $\{f, H\} + \frac{\partial f}{\partial t} = 0$. Here $\{f, H\}$ denotes the Poisson Bracket.

**Symmetries of Hamiltonian systems 5.1.3**

**Infinitesimal Transformation of the Hamiltonian5.1.3.1 :**

A one-parameter continuous group of (possibly time dependent) transformations of the dynamical variables which leave their time evolution invariant and therefore leave the Lagrangian invariant up to total derivative, $\frac{dG(q)}{dt}$ , induces the following infinitesimal transformations of Hamiltonian

$$\delta H = H(q', p', t) - H(q, p, t) = \varepsilon \frac{\partial F}{\partial t} + \varepsilon \frac{dG(q)}{dt}. \tag{26}$$

where $F$ is the (possibly time dependent) generator of the corresponding canonical transformation $q, p \rightarrow q', p'$.

In this more general case, the Hamiltonian formulation of Noether theorem gives the following conservation law

$$\frac{dQ}{dt} = \frac{d}{dt}(F + G) = 0. \tag{27}$$

Hence, in order to get a conserved quantity we have to add the function $G$ to the generator F of the canonical transformations.

Since the addition of a total derivative to the Lagrangian does not change the dynamics of the variables, $q, \dot{q}$, it leaves invariant all the observables $F(q, \dot{q})$ and has therefore the meaning of a gauge transformation.

In terms of the canonical variables the addition of the total derivative implies the following transformation of the canonical variables

$$q_i \rightarrow q_i\ ,\ p_i \rightarrow p_i - \frac{\partial G}{\partial q_i}. \tag{28}$$

which changes the relation between the conjugate momentum $p_i$ and the time derivative of position equation $(28)$ states that $G$ is the canonical generator of such gauge transformation. It is worthwhile to recall that for a particle in a magnetic field $x$ and $\dot{x}$ are observable (gauge invariant ) quantities, but $p_i \equiv \dot{x}_i + (e/c)A_i$ is not.

In conclusion for one- parameter continuous groups of transformations, which leave the dynamics invariant, but leave the Lagrangian or the Hamiltonian invariant only up to a total derivative, the conservation laws displays a sort of anomaly, the conserved quantity being the sum f the generator of the corresponding canonical transformation plus the generator G of the gauge transformation $(28)$.

**Symmetries of the Dynamics and Transformation of the Hamiltonian 5.1.3.2 :**
The symmetry of the dynamics we mean a transformation of the dynamical variables such that their equations of motion are invariant. In the Lagrangian formulation, the dynamical variables are the Lagrangian coordinates $q_i$ and their time derivatives $\dot{q}_i$ and a transformation.

$$q_i \to q_i^{'}(q,t), \qquad \dot{q}_i \to \dot{q}_i^{'}(q,\dot{q},t) \tag{29}$$

is a symmetry of the dynamics if it leaves the equations of motion
$\ddot{q} = F_i(q,\dot{q},t)$ invariant, i.e.

$$\ddot{q}_i{'} = F_i(q',\dot{q}',t). \tag{30}$$

Then, we have a complete characterization of the symmetries of the dynamics in terms of invariance properties of the Lagrangian.

**Proposition 5.1.4 :**
The invariance of the Lagrange equations under a (possible time dependent) transformation of the Lagrangian variables $q_i \to q_i^{'}$, $\dot{q}_i \to \dot{q}_i^{'}$ is equivalent to the invariance of the Lagrangian up to a total derivative

$$L'(q',\dot{q}',t) = L(q',\dot{q}',t) - \frac{dG(q)}{dt}. \tag{31}$$

Since the Lagrangian transforms covariantly under a change of the Lagrangian coordinates, namely

$$L'(q',\dot{q}',t) = L(q,\dot{q},t). \tag{32}$$

Eq. $(H0)$ (and therefore the symmetry of the dynamics) is equivalent to

$$L(q',\dot{q}',t) = L(q,\dot{q},t) + \frac{dG(q)}{dt}. \tag{33}$$

**5.2 Lagrangian mechanics**

Suppose a system is defined by Lagrangian $L$ with generalized coordinates . If $L$ has no explicit time dependence (so $\frac{\partial L}{\partial t} = 0$) , then the energy $E$ defined by

$$E = \sum_i \left[ \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right] - L \tag{34}$$

Is conserved .

Furthermore , if $\frac{\partial L}{\partial q} = 0$ , then $q$ is said to be cyclic coordinate and the generalized momentum $p$ defined by

$$p = \frac{\partial L}{\partial \dot{q}} \tag{35}$$

Is conserved .

**5.3Symmetries of Lagrangian systems**

Invariance of the given variational integral under a group of symmetries implies that the associated Euler-Lagrange equations is also invariant under the group.

**Theorem 5.3.1:**

If*G is a variational symmetry group of the functional* $\ell[u] = \int_{\Omega_0} L\left(x, u^{(n)}\right) dx$ , then*G is a symm*

*group of the Euler-Lagrange equations* $E(L) = 0$.

Intuitively, what is happening is that if $g \in G$ and $u = f(x)$ is anextremal of $\ell[u]$, then clear

$\tilde{u} = g \cdot f(\tilde{x})$ is an extremal of the transformed variational problem $\tilde{\ell}[\tilde{u}]$ coming from

$(\tilde{x}, \tilde{u}) = g \cdot (x, u)$.

But if $G$ is a variational symmetry group, $\tilde{\ell}[\tilde{u}] = \ell[\tilde{u}]$, hence $g \cdot f$ is also an extremal of

It is *not* true that every symmetry group of the Euler-Lagrange equations is also a variational symmetry group of the original variational problem! The most common counterexamples are given by groups of scaling transformations.

**6. Invariant:**

**6.1 Definition:** If $F: M \to N$ is a diffeomorphismand $X$ is a $C^\infty$ vector field on $M$ such $F_*(X) = X$ . that is, $X$ is F-related to it- self, then $X$ is said to be invariant with respect to $F$ - invariant.

**6.2 Remark :**

Let us define two diffeomorphism to Lie group called left translation $L_a$ and right translation by anddefined by.

$$L_a : G \to G \ , \ L_a g = ag . \quad R_a : G \to G \quad , \quad R_a g = ag .$$

**6.3 Action of Groups** :

Important groups action is the following actions of $G$ on itself. Left action: $L_g : G \to G$ is defined by $L_g(h) = gh$ .

Right action: $R_g : G \to G$ is defined by $R_g(h) = hg^{-1}$.

Adjoint action: $Ad_g : G \to G$ is defined by $Ad_g(h) = ghg^{-1}$.

Easily sees that left and right actions are transitive; in fact, each of them is simply transitive. also easy to see that left and right actions are commute and that

$$Ad_g = L_g R_g .$$

Each of these actions also defines the action of $G$ on the spaces of functions, vector fields, fo etc. On $G$ .

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**6.4 Definition:**

Avector field $v \in Vect(G)$ is left- invariant if $g \cdot v = v$ for every, $g \in G$ and right -invariant if $v \cdot g = v$ **,** for every $g \in G$. A vector field is called bi-invariant if it is both left- and right-invariant.

**6.5 Definition (Left invariant vector field)**

By using left translations to transport around tangent vectors on $G$. put $g_1 = T_e G$ , The tangent space to $G$ at the neutral element $e \in G$ . For $X \in g_1$ and $g \in G$ define.

$$L_x(g) = T_e \lambda_g . \qquad X \in T_g G.$$

**6.6 Definition:**

Let $G$ is a Lie group. A vector field $\xi \in \chi(G)$ is called left invariant if and only if $(\lambda_g) * \xi = \xi$ for all $g \in G$. The space of left invariant vector fields is denoted by $\chi_L(G)$.

**6.7 Definition: (Right invariant vector fields)**

We have used left translations to trivialize the tangent bundle of a Lie group G ; one can consider the right trivialization. $TG \to G \times g_1$ Defined by

$\xi_g \to \left(g , T_g \rho^{g^{-1}} \cdot \xi\right)$**.** The inverse of this map is denoted by $(g, X) \to R_X(g)$, and $R_X$ is called the right - invariant vector field generated by $X \in g$. In general, a vector field $\xi \in \chi(G)$ is called right invariant if $(\rho^g)^* \xi = \xi$ for all $g \in G$ . The space of right invariant vector fields is denoted by $\chi_R(G)$.

$\xi = \xi(e)$ and $X \to R_X$. Are inverse bijections between g and $\chi_R(G)$.

**6.8 Proposition :**

Let $G$ be a Lie group, then, the vector space of all left -invariant vector fields on $G$ is isomorphic( as a vector space) to $T_1 G$.

$\Gamma(TG)^G \cong T_1 G$ .

We now give $T_1 G$ a Lie algebra structure by identifying if with $\Gamma(TG)^G$ with the lie bracket of vector fields. But, we need to show at $[,]$ is in fact a binary operation on $\Gamma(TG)^G$ , recall if $f : M \to N$ is a smooth map of manifolds and $X, Y$ are $f$ – related if $d(X(x)) = Y(f(x))$ for every $x \in M$ . It is fact from manifolds theory that is $X, Y$ and $X', Y'$ are $F$ – related , then so are $[X, Y]$ and $[X', Y']$, but, left invariant vector fields are $L_a$ related for all $a \in G$ by definition.

**6.9 Proposition :**

The Lie bracket of two left vector fields is a left invariant vector field. Thus we can $T_1 G$ regard as Lie algebra.

**6.10 Definition:** Let $G$ be a group, the Lie algebra g of $G$ is $T_1G$ with the Lie bracket induced by its identification with $\Gamma(TG)^G$ **.**

**6.11 Corollary:** Suppose $\ell[u] = \int L\left(x, u^{(1)}\right)dx$ is afirst order variational problem, and $v$ as

$$v = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha} \qquad (36)$$

A variational symmetry. Then

$$p_i = \sum_{\alpha=1}^{q} \phi_\alpha \frac{\partial L}{\partial u_i^\alpha} + \xi^i L - \sum_{\alpha=1}^{q}\sum_{j=1}^{p} \xi^j u_j^\alpha \frac{\partial L}{\partial u_i^\alpha}, \qquad i=1,\dots,p, \qquad (37)$$

Form the components of a conservation law $Div\ P = 0$ for the Euler-Lagrange equations $E(L) = 0$.

**6.12 Example:**Considera system of $n$ particles moving in $\mathbb{R}^3$ subject to some potential force field. The kinetic energy of this system takes the form

$$K(\dot{x}) = \frac{1}{2}\sum_{\alpha=1}^{n} m_\alpha \left|\dot{x}^\alpha\right|^2,$$

Where $m_\alpha$ is the mass and $x^\alpha = (x^\alpha, y^\alpha, z^\alpha)$ the position of the $\alpha-$th particle.

The potential energy $U(t,x)$ will depend on the specific problem ; for instance ,

$$U(t,x) = \sum \gamma_{\alpha\beta}\left|x^\alpha - x^\beta\right|^{-1}$$

Might depend only on the pairwise gravitational interaction between masses, or if $(n=1)$ we may have the central gravitational force of Kepler's problem.

Newton's equations of motion

$$m_\alpha x_{tt}^\alpha = -\nabla_\alpha U \equiv -\left(U_{x^\alpha}, U_{y^\alpha}, U_{z^\alpha}\right), \qquad \alpha=1,\dots,n,$$

Are in variational form, being the Euler-Lagrange equations for the action integral $\int_{-\infty}^{\infty}(K-U)dt$.

A vector field

$$v = \tau(t,x)\frac{\partial}{\partial t} + \sum_\alpha \xi^\alpha(t,x)\cdot\frac{\partial}{\partial x^\alpha} \equiv \tau\frac{\partial}{\partial t} + \sum_\alpha\left(\xi^\alpha\frac{\partial}{\partial x^\alpha} + \eta^\alpha\frac{\partial}{\partial y^\alpha} + \zeta^\alpha\frac{\partial}{\partial z^\alpha}\right)$$

Will generate a variational symmetry group if and only if

$$pr^{(1)}v(K-U) + (K-U)D_t\tau = 0 \qquad (38)$$

For all $(t,x)$. Noether's theorem immediately provides a corresponding conservation low or first integral

$$T = \sum_{\alpha=1}^{n} m_\alpha \xi^\alpha \cdot \dot{x}^\alpha - \tau E = \text{constant} \ (39)$$

Where $E = K + U$ is the total energy of the system.

## Conclusion:

The aim of this paper is to study symmetries of Lagrangian and Hamiltonian systems using L algebra of the symmetry Lie groups.

The importance of this paper is that one can determine the invariants of a symmetrical syste using the Lie algebra of its symmetry Lie group. This has been illustrated via some examples.

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