Various Characterizations of Optimal Sobolev trace embeddings

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Abstract:

We exhibit and describe optimal target spaces in orbitrary-order Sobolev type embeddings for traces of nn-dimensional functions on lower dimensional subspaces, this method is very efficient to give various characterizations of optimal Sobolev trace embeddings, we used the descriptive-deductive method, Results :we found that any trace embedding can be reduced to a one-dimensional inequality for a Hardy type operator depending only on nn and on the dimension of the relevant subspace.

Keywords: Sobolev spaces, trace embeddings, optimal target, rearrangement-invariant spaces, Orlicz spaces, Lorentz spaces,supremum operators.

توصيفات مختلفة لأثر أمثلية تضمينات سوبوليف

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المستخلص:

الهدف من هذه الدراسة هوعرض ووصف أمثلية الفضاءات المستهدفة في الترتيب المحكم لنمط تضمينات سوبوليف لأثار للدوال ذات البعد-n على البعد الأدنى للفضاءات الجزئية ،هذه الطريقة فعالة جدا في إعطاء توصيفات مختلفة لإثر أمثلية تضمينات سوبوليف ،إستخدمنا المنهج الوصفي الإستنباطي، النتائج :إن أي أثر للتضمين يمكن أن يكون مخفض للمتباينة ذات البعد الواحد لنمط مؤثر هاردي إعتماد فقط على n و على بعد الفضاء الجزئي ذو الصلة.

كلمات مفتاحية: فضاءات سوبوليف، أثرتضمينات،هدف أمثل ، إعادة ترتيب الفضاءات الثابتة، فضاءات أورليش، فضاءات لورنتز، مؤثر أعلى

1. Introduction and main results

Let $\lambda\lambda$ be an open set in $\mathbb{R}^n,\mathbb{R}^n$, with $n \geq 2n \geq 2$and let $d \in \mathbb{N}d \in \mathbb{N}$ such that $1 \leq d \leq n.1 \leq d \leq n$. We denote by λ_d λ_d the (non empty) intersection of $\lambda\lambda$ with a dd-dimensional affine subspace of $\mathbb{R}^n\mathbb{R}^n$. Moreover, given any $m \in \mathbb{N}m \in \mathbb{N}$ and $0 \leq \epsilon \leq \infty 0 \leq \epsilon \leq \infty$, we call $W^{(2+\epsilon,1+\epsilon)}(\lambda)$ $W^{(2+\epsilon,1+\epsilon)}(\lambda)$ the standard Sobolev space of those functions which belong to $L^{(1+\epsilon)}(\lambda)L^{(1+\epsilon)}(\lambda)$ together with all their weak derivatives up to the order mm. If $\lambda\lambda$ is bounded and satisfies the cone condition, and

$$d \geq n - m, \qquad (1.1)$$

then a linear trace operator

$$Tr: W^{(m,1)}(\lambda) \to L^1(\lambda_d) \qquad (1.2)$$

is customarily well defined at any function in $W^{(m,1)}(\lambda)W^{(m,1)}(\lambda)$ via approximation by smooth functions. Here, $L^1(\lambda_d)L^1(\lambda_d)$ stands for a Lebesgue space on $\lambda_d\lambda_d$ with respect to the dd-dimensional Hausdorff measure $\mathcal{H}^d,\mathcal{H}^d$, and the arrow” $\to\to$” denotes a

bounded operator. Of course, if $d = n, d = n$, then $\lambda_d = \lambda, \lambda_d = \lambda$, and $TrTr$ is the identity operator.
An even sharper version of the trace embedding (1.2) is available for the space $W^{(m,1+\epsilon)}(\lambda)W^{(m,1+\epsilon)}(\lambda)$ for every $m \in \mathbb{N} m \in \mathbb{N}$ and $\epsilon \geq 0, \epsilon \geq 0$, and reads

$$Tr: W^{(m,1+\epsilon)}(\lambda) \to \begin{cases} L^{\frac{(1+\epsilon,d)}{n-m(1+\epsilon)}}(\lambda_d) \text{ if } \epsilon < \frac{-1}{2} & \text{and } \quad -\frac{1}{2} \leq \epsilon < \frac{1}{2}, \\ \exp L^{\frac{n}{n-m}}(\lambda_d) \ \text{ if } \epsilon < \frac{-1}{2} & \text{and } \ \epsilon = 0 \text{ or } \epsilon = \frac{-3}{2}, \\ L^{\infty}(\lambda_d) & \text{otherwise,} \end{cases} \quad (1.3)$$

where $\exp L^{\frac{n}{n-m}}(\lambda_d) \exp L^{\frac{n}{n-m}}(\lambda_d)$ denotes an Orlicz space of exponentially integrable functions on $\lambda_d . \lambda_d$. Equation (1.3) collects classical embedding theorems due to [20] ($-\frac{1}{2} \leq \epsilon < \frac{1}{2} - \frac{1}{2} \leq \epsilon < \frac{1}{2}$ or $\epsilon > 0, \epsilon > \frac{-3}{2} \epsilon > 0, \epsilon > \frac{-3}{2}$) ([27] Nirenberg,1959]($\epsilon = 0, d = n$ $\epsilon = 0, d = n$), [33] ($d = n, d = n$, and($-\frac{1}{2} < \epsilon < \frac{1}{2} - \frac{1}{2} < \epsilon < \frac{1}{2}$ or $\epsilon > 0$ or $\epsilon > \frac{-3}{2} \epsilon > 0$ or $\epsilon > \frac{-3}{2}$), [32] ($d = 1 - \epsilon, \epsilon = 0$ or $\epsilon = \frac{-3}{2}$ $d = 1 - \epsilon, \epsilon = 0$ or $\epsilon = \frac{-3}{2}$), [1] ($\epsilon = 0$ or $\epsilon = \frac{-3}{2} \epsilon = 0$ or $\epsilon = \frac{-3}{2}$). In order to explain the aims of the present section, which deals with a general class of Sobolev type trace embeddings, let us focus for a moment on the specific instance (1.3), and recall that, in the case when $d = 1 - \epsilon, d = 1 - \epsilon$, the target space in all such embeddings is optimal in the class of Lebesgue spaces $-\frac{1}{2} \leq \epsilon < \frac{1}{2}$ $-\frac{1}{2} \leq \epsilon < \frac{1}{2}$ or $\epsilon > 0$ or $\epsilon > \frac{-3}{2} \epsilon > 0$ or $\epsilon > \frac{-3}{2}$), and also Orlicz spaces

($\epsilon \geq 0 \epsilon \geq 0$) [10], but can be improved (for $\epsilon \leq 0, \epsilon \leq \frac{-3}{2}$

$\epsilon \le 0, \epsilon \le \frac{-3}{2}$) on replacing the Lebesgue space $L^{\left(\frac{1-\epsilon^2}{3}\right)}(\lambda)$ $L^{\left(\frac{1-\epsilon^2}{3}\right)}(\lambda)$ and the Orlicz space $\exp L^{\left(\frac{1-\epsilon^2}{3}\right)}(\lambda) \exp L^{\left(\frac{1-\epsilon^2}{3}\right)}(\lambda)$ with the strictly smaller Lorentz space $L^{\left(\frac{1-\epsilon^2}{1-\epsilon-2+\epsilon(1+\epsilon)},1+\epsilon\right)}(\lambda)$ $L^{\left(\frac{1-\epsilon^2}{1-\epsilon-2+\epsilon(1+\epsilon)},1+\epsilon\right)}(\lambda)$ [28], [30] and Lorentz-Zygmund space $L^{\left(\infty,\frac{1}{1+2\epsilon};-1\right)}(\lambda)$, $L^{\left(\infty,\frac{1}{1+2\epsilon};-1\right)}(\lambda)$, [5] , [24], respectively.These latter spaces turn out to be optimal (smallest possible) in the class of allrearrangement-invariant spaces on λ,λ, namely, loosely speaking, the Banach spaces of measurable functions on $\lambda\lambda$ endowed with a norm which only depends on integrability properties of functions. A question which thus naturally arises in this regard is whether the target spaces in (1.3) can be enhanced, in a similar spirit, also when genuine trace embeddings are in question, that is when $d \le n-1$. $d \le n-1$. This question can be affirmatively answered from an application of one of the main results of this section, which characterizes the optimal rearrangement-invariant target space in the trace embedding for given Sobolev type domain space built upon any rearrangement-invariant space.Indeed, a specialization of this result tells us that, if $m < n$,$m < n$, then

$$Tr: W^{(m,1+\epsilon)}(\lambda) \to \begin{cases} L^{\frac{(1+\epsilon)d}{n-m(1+\epsilon)},(1+\epsilon)}(\lambda_d) & \text{if } -\frac{1}{2} \le \epsilon < \frac{1}{2}, \\ L^{\left(\infty,\frac{1}{1+2\epsilon};-1\right)}(\lambda_d) & \text{if } \epsilon = 0, \epsilon = \frac{-3}{2} \end{cases}. \tag{1.4}$$

The trace embeddings in (1.4) are, in turn, a special instance of Theorem (5.1), where applications of our approach to optimal trace embeddings for Lorentz-Sobolev and Orlicz-Sobolev spaces are exhibited. Note that the trace embeddings in (1.4) actually improve the first two embeddings in (1.3), since $L^{\left(\frac{(1+\epsilon)d}{n-m(1+\epsilon)},1+\epsilon\right)}(\lambda_d) \subsetneq L^{\left(\frac{(1+\epsilon)d}{n-m(1+\epsilon)}\right)}(\lambda_d)$

$L^{\left(\frac{(1+\epsilon)d}{n-m(1+\epsilon)},1+\epsilon\right)}(\lambda_d) \subsetneq L^{\left(\frac{(1+\epsilon)d}{n-m(1+\epsilon)}\right)}(\lambda_d)$ (unless $\epsilon = 0$ $\epsilon = 0$ and $d = n - m, d = n - m,$ in which case the two spaces coincide), and $L^{\left(\infty,\frac{1}{1+2\epsilon};-1\right)}(\lambda_d) \subsetneq \exp L^{\frac{n}{n-m}}(\lambda_d).$ $L^{\left(\infty,\frac{1}{1+2\epsilon};-1\right)}(\lambda_d) \subsetneq \exp L^{\frac{n}{n-m}}(\lambda_d)$. Moreover, the target spaces in (1.4) are optimal among all rearrangement-invariant spaces.Our general version of the optimal Sobolev trace embedding is stated in Theorem 1.1below, and requires a few preliminaries.Given any rearrangement-invariant function norms $\|\cdot\|_{X(0,1)}\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)},\|\cdot\|_{Y(0,1)},$ we denote by $X(\lambda)X(\lambda)$ the rearrangement-invariant space on $\lambda\lambda$ associated with $\|\cdot\|_{X(0,1)}\|\cdot\|_{X(0,1)}$ and by $Y(\lambda_d)$ $Y(\lambda_d)$ the rearrangement-invariant space on $\lambda_d,\lambda_d,$ with respect to the dd-dimensional Hausdorff measure $\mathcal{H}^d\mathcal{H}^d$ restricted to $\lambda_d,\lambda_d,$ associated with $\|\cdot\|_{Y(0,1)}.\|\cdot\|_{Y(0,1)}$. We then call $W^mX(\lambda)W^mX(\lambda)$ the Sobolev type Banach space of all functions which belong to $X(\lambda)$ $X(\lambda)$ together with all their weak derivatives up to the order mm . Hence, $W^mX(\lambda) = W^{(m,1+\epsilon)}X(\lambda)W^mX(\lambda) = W^{(m,1+\epsilon)}X(\lambda)$ if $X(\lambda) = L^{(1+\epsilon)}(\lambda). X(\lambda) = L^{(1+\epsilon)}(\lambda)$. We also denote by $W_\perp^mX(\lambda)$ $W_\perp^mX(\lambda)$ the subspace of those functions from $W^mX(\lambda)W^mX(\lambda)$ whose mean value over $\Omega\Omega$ is 00, together with the mean value of all their weak derivatives up to the order $(m-1).(m-1).$ Let us briefly comment on assumption (1.1). Since $X(\lambda) = L^1(\lambda)$ $X(\lambda) = L^1(\lambda)$ for any rearrangement-invariant space provided that $\lambda\lambda$ has finite measure, one has that $W^mX(\lambda) \to W^{m,1}X(\lambda)$ $W^mX(\lambda) \to W^{m,1}X(\lambda)$ for any $m \in \mathbb{N} m \in \mathbb{N}$ and any such space $X(\lambda).X(\lambda)$. Thus, by (1.2), under assumption (1.1) the trace operator $TrTr$ is certainly well defined from $W^mX(\lambda)W^mX(\lambda)$ into $L^1X(\lambda_d)L^1X(\lambda_d)$ (at least), whatever $m \in \mathbb{N}m \in \mathbb{N}$ and $X(\lambda)$ $X(\lambda)$ are.On the other hand, dropping this assumption (in the case when $m < n\,m < n$) would exclude Sobolev type spaces built upon rearrangement-invariant spaces $X(\lambda)X(\lambda)$ endowed with a too weak norm,for instance $L^1(\lambda).L^1(\lambda)$. Since we are not going to impose any restriction on the rearrangement-invariant space $X(\lambda)X(\lambda)$ in the main results of the section, we shall keep (1.1) in force throughout. Now, given $n, m, d \in \mathbb{N}n, m, d \in \mathbb{N}$ such that $1 \le d \le n1 \le d \le n$ and $d \ge n-m, d \ge n-m,$ we call $\|\cdot\|_{X^m_{d,n}(0,1)}\|\cdot\|_{X^m_{d,n}(0,1)}$ the rearrangement-invariant function norm whose associate function norm is given by

$$\left\|\sum_j f_j\right\|_{(X^m_{d,n})'(0,1)} = \left\|(1+2\epsilon)^{-1+\frac{m}{n}}\int_0^{(1+2\epsilon)^{\frac{d}{n}}}\sum_j f_j^*(r)dr\right\|_{X'(0,1)} \qquad (1.5)$$

for every nonnegative measurable function f_jf_j on $(0,1).(0,1).$

Theorem1.1. [Optimal target spaces for trace embeddings] Let $\lambda\lambda$ be a bounded open set with the cone property in $\mathbb{R}^n, n \ge 2.$ $\mathbb{R}^n, n \ge 2.$ Assume that $m \in \mathbb{N}m \in \mathbb{N}$ and $d \in \mathbb{N}d \in \mathbb{N}$ are such that $1 \le d \le n\;1 \le d \le n$ and $d \ge n-m, d \ge n-m,$ and let $\|\cdot\|_{X(0,1)}\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Let $\|\cdot\|_{X^m_{d,n}(0,1)}\|\cdot\|_{X^m_{d,n}(0,1)}$ be the rearrangement-invariant function norm obeying (1.5).Then

$$Tr: W^mX(\lambda) \to X^m_{d,n}(\lambda_d). \qquad (1.6)$$

Moreover, the space $X^m_{d,n}(\lambda_d)X^m_{d,n}(\lambda_d)$is optimal in (1.6) among all rearrangement-invariant spaces. An important special case of

Theorem 1.1 is enucleated in Corollary 1.2, which provides us with a characterization of the a Sobolev spaces $W^m X(\lambda) W^m X(\lambda)$ which are mapped into $L^\infty(\lambda_d) L^\infty(\lambda_d)$ be the trace operator [19].

Corollary 1.2. [Trace embedding into $L^\infty L^\infty$] Let $n, d, m, \lambda,$ $n, d, m, \lambda,$ and $\|\cdot\|_{X(0,1)} \|\cdot\|_{X(0,1)}$ be as in Theorem 1.1. Then the following facts are equivalent:

$$Tr: W^m X(\lambda) \to L^\infty(\lambda_d) \qquad (1.7)$$

$$X^m_{d,n}(\lambda_d) = L^\infty(\lambda_d) \qquad (1.8)$$

$$\left\| (1+2\epsilon)^{-1+\frac{m}{n}} \right\|_{X'(0,1)} < \infty. \qquad (1.9)$$

In particular,(1.7) and (1.8) hold for any rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}, \|\cdot\|_{X(0,1)},$ provided that $m \geq n.$ $m \geq n.$
The proof of Theorem 1.1 relies upon a reduction principle for trace embeddings, and corresponding Poincaré trace inequalities, ensuring that any such embedding is equivalent to a one-dimensional inequality for suitable Hardy type operator.This is the content of the main result of this section.

Theorem 1.3. [Reduction principle for trace embeddings] Let $n, d, m, \lambda, n, d, m, \lambda,$ be as in Theorem 1.1.Let $\|\cdot\|_{X(0,1)} \|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)} \|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Then the following facts are equivalent.

(i) The Sobolev trace embedding

$$Tr: W^m X(\lambda) \to Y(\lambda_d) \qquad (1.10)$$

holds.

(ii) The Poincaré trace inequality

$$\|Tr\, u\|_{Y(\lambda_d)} \leq C_1 \|\nabla^m u\|_{X(\lambda)} \qquad (1.11)$$

holds for some constant $C_1, C_1,$ and for every $u \in W^m_\perp X(\lambda).$ $u \in W^m_\perp X(\lambda).$

(iii) The inequality

$$\left\| \int_{(1-2\epsilon)^{\frac{n}{d}}}^{1} \sum_j f_j (1+2\epsilon)(1+2\epsilon)^{-1+\frac{m}{n}} d(1+2\epsilon) \right\|_{Y(0,1)}$$
$$\leq C_2 \sum_j \|f_j\|_{X(0,1)} \qquad (1.12)$$

holds for some constant $C_2, C_2,$ and for every nonnegative $f_j \in X(0,1) f_j \in X(0,1)$.

Theorem 1.4. [Sharp iteration principle for trace embeddings] Let $\lambda\lambda$ be an open set with the cone property in $\mathbb{R}^n, n \geq 2. \mathbb{R}^n, n \geq 2.$ Let $k, h, d, \ell \in \mathbb{N} k, h, d, \ell \in \mathbb{N}$ be such that $1 \leq d \leq \ell \leq n, \ell \geq n-k$ $1 \leq d \leq \ell \leq n, \ell \geq n-k$ and $d \geq \ell - h. d \geq \ell - h.$ Assume that $\lambda_d \subset \lambda_\ell . \lambda_d \subset \lambda_\ell$. Let $\|\cdot\|_{X(0,1)} \|\cdot\|_{X(0,1)}$ be rearrangement-invariant function norm.Then

$$\left(X_{\ell,n}^{k}\right)_{d,\ell}^{h} (\lambda_d) = X_{d,n}^{k+h} (\lambda_d). \qquad (1.13)$$

The validity of the iteration principle of Theorem1.4 in such a full generality is quite striking. In fact, although iteration of Sobolev and or trace embeddings,with optimal targets in subclasses of rearrangement-invariant spaces, may yield sharp higher-order results in customary settings (see e.g. ([22], [37]), this is not always the case, especially when borderline situations are in question.To verify this assertion, consider, for instance, the third embedding in (1.3), with $m = n m = n$ and $\epsilon = 0 \epsilon = 0$, namely $Tr: W^{n,1}(\lambda) \to L^\infty(\lambda_d). Tr: W^{n,1}(\lambda) \to L^\infty(\lambda_d).$

If $2 \leq d \leq n, 2 \leq d \leq n,$ this trace embedding cannot be recovered from a subsequent application of the first embedding in (1.3) with $m = n-1 m = n-1$ and of the second embedding in (1.3) with $m = 1 m = 1$ and $d = n d = n$, since this argument only yields the weaker conclusion

$$W^{n,1}(\lambda) \xrightarrow{Tr} W^{1,d}(\lambda_d) \to \exp L^{\frac{d}{d-1}}(\lambda_d). \qquad (1.14)$$

Notice that both the trace and the Sobolev embedding in (1.14) are optimal within the class of Orlicz spaces.The following example is even more enlightening, in that it illustrates a variety of situations which can occur after iteration of Sobolev trace embeddings. Assume that $n \geq 3, m = 2, n \geq 3, m = 2,$ and $\max\{2, n-2\} \leq d \leq n.\max\{2, n-2\} \leq d \leq n.$ By the second trace embedding in (1.3), with $1+\epsilon = \frac{n}{2}, 1+\epsilon = \frac{n}{2},$

$$Tr: W^{2,\frac{n}{2}}(\lambda) \to \exp L^{\frac{n}{n-2}}(\lambda_d). \qquad (1.15)$$

Appropriate choices of mm and dd in the first two trace embeddings in (1.3) yield

$$W^{2,\frac{n}{2}}(\lambda) \xrightarrow{Tr} W^{1,d}(\lambda_d) \to \exp L^{\frac{d}{d-1}}(\lambda_d). \qquad (1.16)$$

and

$$W^{2,\frac{n}{2}}(\lambda) \to W^{1,n}X(\lambda) \xrightarrow{Tr} \exp L^{\frac{n}{n-1}}(\lambda_d). \qquad (1.17)$$

Observe that

$$\exp L^{\frac{n}{n-2}}(\lambda_d) \subseteq \exp L^{\frac{d}{d-1}}(\lambda_d) \subseteq \exp L^{\frac{n}{n-1}}(\lambda_d), \qquad (1.18)$$

where the first inclusion is strict whenever $(n,d) \neq (4,2)$, $(n,d) \neq (4,2)$, and the second inclusion is strict if $d < nd < n$. Thus, for these values of $n\, n$ and $d\, d$, equations (1.15), (1.16) and (1.17) yield different results, although all of them are obtained from compositions of trace embeddings with optimal Orlicz targets. Futhermore, only the first one has a resulting optimal Orlicz target. On the other hand, in the special case when $n = 4$ $n = 4$ and $d = 2d = 2$, the first inclusion in (1.18) is in fact an identity, and hence the composition in (1.16) does yield a trace embedding with optimal eventual Orlicz target.

2. Function spaces

2.1. Spaces of measurable functions we shall now briefly recall some basic facts from the theory of rearrangement-invariant spaces (see [4]).

Let $(\mathcal{R},\nu)(\mathcal{R},\nu)$ be a finite positive measure space.We denote by $\mathcal{M}(\mathcal{R},\nu)\mathcal{M}(\mathcal{R},\nu)$ be the set of all $\nu\nu$-measurable functions on $\mathcal{R}\mathcal{R}$ taking values in $[-\infty,\infty].[-\infty,\infty]$. We also defined $\mathcal{M}_+(\mathcal{R},\nu)=\{u\in\mathcal{M}(\mathcal{R},\nu): u\geq 0\}$ $\mathcal{M}_+(\mathcal{R},\nu)=\{u\in\mathcal{M}(\mathcal{R},\nu): u\geq 0\}$ and $\mathcal{M}_0(\mathcal{R},\nu)=\{u\in\mathcal{M}(\mathcal{R},\nu): u \text{ is finite } \nu-\text{a. e. on } \mathcal{R}\}.$ $\mathcal{M}_0(\mathcal{R},\nu)=\{u\in\mathcal{M}(\mathcal{R},\nu): u \text{ is finite } \nu-\text{a. e. on } \mathcal{R}\}.$ If $\mathcal{R}$ $\mathcal{R}$ is a subset of $\mathbb{R}^n\mathbb{R}^n$ equipped with the Lebesgue measure, then $(\mathcal{R},\nu)(\mathcal{R},\nu)$ will be simply denoted by $\mathcal{R}.\mathcal{R}.$Given any function $u\in\mathcal{M}(\mathcal{R},\nu),u\in\mathcal{M}(\mathcal{R},\nu)$, its non-increasing rearrangement $u^*:[0,\infty)\to[0,\infty]u^*:[0,\infty)\to[0,\infty]$ is defined as

$$u^*(1+2\epsilon)=\sup\{1-2\epsilon\in\mathbb{R}:\nu(\{x\in\mathcal{R}:|u(x)|>1-2\epsilon\})>1+2\epsilon\}$$
$$u^*(1+2\epsilon)=\sup\{1-2\epsilon\in\mathbb{R}:\nu(\{x\in\mathcal{R}:|u(x)|>1-2\epsilon\})>1+2\epsilon\}$$

for $\frac{-1}{2}\leq\epsilon<\infty\frac{-1}{2}\leq\epsilon<\infty$.We also define $u^{**}:(0,\infty)\to[0,\infty]$ $u^{**}:(0,\infty)\to[0,\infty]$ as $u^{**}(1+2\epsilon)=\frac{1}{1+2\epsilon}\int_0^{1+2\epsilon}u^*(r)dr$ $u^{**}(1+2\epsilon)=\frac{1}{1+2\epsilon}\int_0^{1+2\epsilon}u^*(r)dr$ for $\frac{-1}{2}<\epsilon<\infty$ $\frac{-1}{2}<\epsilon<\infty$. Note that $u^{**}u^{**}$is also non-increasing, and $u^*(1+2\epsilon)\leq u^{**}(1+2\epsilon)u^*(1+2\epsilon)\leq u^{**}(1+2\epsilon)$ for $\frac{-1}{2}<\epsilon<\infty\frac{-1}{2}<\epsilon<\infty$. Moreover,

$$\int_0^{1-2\epsilon} (u+v)^*(1+2\epsilon)d(1+2\epsilon)$$
$$\leq \int_0^{1-2\epsilon} u^*(1+2\epsilon)d(1+2\epsilon)$$
$$+\int_0^{1-2\epsilon} v^*(1+2\epsilon)d(1+2\epsilon) \qquad (2.1)$$

for $\frac{1}{2} \geq \epsilon > \infty \frac{1}{2} \geq \epsilon > \infty$, for every $u, v \in \mathcal{M}_+(\mathcal{R}, \nu)$. $u, v \in \mathcal{M}_+(\mathcal{R}, \nu)$.Two measurable functions uu and vv on SS are said to be equimeasurable (or equidistributed) if $u^* = v^*.u^* = v^*$. A basic property of rearrangements is the Hardy-Littlewood inequality,which tell us that, if $u, v \in \mathcal{M}(\mathcal{R}, \nu), u, v \in \mathcal{M}(\mathcal{R}, \nu)$, then

$$\int_{\mathcal{R}} |u(x)v(x)|\, dv(x) \leq \int_0^{v(\mathcal{R})} u^*(1+2\epsilon)v^*(1+2\epsilon)d(1+2\epsilon). \qquad (2.2)$$

We say that a functional $\|\cdot\|_{X(0,1)}: \mathcal{M}_+(0,1) \to [0,\infty]$ $\|\cdot\|_{X(0,1)}: \mathcal{M}_+(0,1) \to [0,\infty]$ is a function norm, if for all f_j, g_j f_j, g_j and $\{f_n\}_{n\in\mathbb{N}} \in \mathcal{M}_+(0,1)\{f_n\}_{n\in\mathbb{N}} \in \mathcal{M}_+(0,1)$ and every $0 \leq a < \infty, 0 \leq a < \infty$, the following properties hold :

(P1) $\|\sum_j f_j\|_{X(0,1)} = 0 \|\sum_j f_j\|_{X(0,1)} = 0$ if and only if $f_j = 0$; $\|a\sum_j f_j\|_{X(0,1)} = a\|\sum_j f_j\|_{X(0,1)}$; $\|\sum_j f_j + g_j\|_{X(0,1)} \leq \sum_j \|f_j\|_{X(0,1)} + \sum_j \|g_j\|_{X(0,1)}$ $f_j = 0$; $\|a\sum_j f_j\|_{X(0,1)} = a\|\sum_j f_j\|_{X(0,1)}$; $\|\sum_j f_j + g_j\|_{X(0,1)} \leq \sum_j \|f_j\|_{X(0,1)} + \sum_j \|g_j\|_{X(0,1)}$;

(P2) $f_j \leq g_j f_j \leq g_j$ a.e. implies $\|\sum_j f_j\|_{X(0,1)} \leq \sum_j \|g_j\|_{X(0,1)}$

$\|\sum_j f_j\|_{X(0,1)} \leq \sum_j \|g_j\|_{X(0,1)}$;

(P3) $f_n \nearrow f_j \quad f_n \nearrow f_j$ a.e. implies $\|f_n\|_{X(0,1)} \nearrow \|\sum_j f_j\|_{X(0,1)}$ $\|f_n\|_{X(0,1)} \nearrow \|\sum_j f_j\|_{X(0,1)}$;

(P4) $\|1\|_{X(0,1)} < \infty \|1\|_{X(0,1)} < \infty$;

(P5) a constant CC exists such that

$\int_0^1 \sum_j f_j(x)\, dv(x) \leq C \sum_j \|f_j\|_{X(0,1)}$.

$\int_0^1 \sum_j f_j(x)\, dv(x) \leq C \sum_j \|f_j\|_{X(0,1)}$.

If, in addition,

(P6) $\|\sum_j f_j\|_{X(0,1)} = \|\sum_j g_j\|_{X(0,1)} \|\sum_j f_j\|_{X(0,1)} = \|\sum_j g_j\|_{X(0,1)}$

whenever $f_j^* = g_j^*, f_j^* = g_j^*$,

we say that $\|\cdot\|_{X(0,1)} \|\cdot\|_{X(0,1)}$ is a rearrangement-invariant function norm. Given a function norm $\|\cdot\|_{X(0,1)}, \|\cdot\|_{X(0,1)}$, we introduce another functional on $\mathcal{M}_+(0,1) \mathcal{M}_+(0,1)$, denoted by $\|\cdot\|_{X'(0,1)}$ $\|\cdot\|_{X'(0,1)}$ and defined as

$$\left\|\sum_j f_j\right\|_{X'(0,1)} = \sup_{g_j \in \mathcal{M}_+(0,1) \|\sum_j g_j\|_{X(0,1)}} \int_0^1 \sum_j f_j(x)\, g_j(1+2\epsilon)\, d(1+2\epsilon).$$

Then $\|\cdot\|_{X'(0,1)} \|\cdot\|_{X'(0,1)}$ is also a function norm on $\mathcal{M}_+(0,1)$. $\mathcal{M}_+(0,1)$.We shall call it the associate norm of $\|\cdot\|_{X(0,1)} \|\cdot\|_{X(0,1)}$. Note that

$$\|\cdot\|_{(X')'(0,1)} = \|\cdot\|_{X(0,1)} \quad . (2.3)$$

Let $\|\cdot\|_{X(0,1)} \|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Then the space $X(\mathcal{R}, \nu) X(\mathcal{R}, \nu)$ is defined as the collection of all functions $u \in \mathcal{M}(\mathcal{R},\nu) u \in \mathcal{M}(\mathcal{R},\nu)$ such that the quantity

$$\|u\|_{X(\mathcal{R},\nu)} = \|u^*(v(\mathcal{R})(1+2\epsilon))\|_{X(0,1)} \qquad (2.4)$$

is finite. The space $X(\mathcal{R},\nu) X(\mathcal{R},\nu)$ is a Banach space, endowed with the norm given by (2.4). With abuse of notation, if $\mathcal{R}=(0,1)$ $\mathcal{R}=(0,1)$ and vv is the Lebesgue measure, we denote $X(\mathcal{R},\nu)$ $X(\mathcal{R},\nu)$ simply by $X(0,1).X(0,1)$. The space $X(0,1)X(0,1)$ is called the representation space of $X(\mathcal{R},\nu).X(\mathcal{R},\nu)$. Given a rearrangement-invariant space $X,X,$the rearrangement-invariant space $X'(\mathcal{R},\nu)X'(\mathcal{R},\nu)$ built upon the function norm$\|\cdot\|_{X'(0,1)}$ $\|\cdot\|_{X'(0,1)}$ is called the associate space of $X(\mathcal{R},\nu).X(\mathcal{R},\nu)$. It turns out that $X''(\mathcal{R},\nu)=X(\mathcal{R},\nu);X''(\mathcal{R},\nu)=X(\mathcal{R},\nu);$ hence, any rearrangement-invariant space $X(\mathcal{R},\nu)X(\mathcal{R},\nu)$ is always the associate space of another rearrangement-invariant space $X'(\mathcal{R},\nu).$ $X'(\mathcal{R},\nu)$. Futhermore, the Hölder inequality

$$\int_0^1 \sum_j f_j(1-2\epsilon)g_j(1-2\epsilon)d(1-2\epsilon) \le \sum_j \|f_j\|_{X(0,1)}\|g_j\|_{X'(0,1)}$$

$$\int_0^1 \sum_j f_j(1-2\epsilon)g_j(1-2\epsilon)d(1-2\epsilon) \le \sum_j \|f_j\|_{X(0,1)}\|g_j\|_{X'(0,1)}$$

holds for every $f_j, g_j \in \mathcal{M}_+(0,1) f_j, g_j \in \mathcal{M}_+(0,1),$ and hence $\frac{1}{v(R)}\int_{\mathcal{R}} |u(x)v(x)|\,dv(x) \le \|u\|_{X(\mathcal{R},\nu)}\|v\|_{X'(\mathcal{R},\nu)}$

$$\frac{1}{v(R)}\int_{\mathcal{R}} |u(x)v(x)|\,dv(x) \le \|u\|_{X(\mathcal{R},\nu)}\|v\|_{X'(\mathcal{R},\nu)}$$

for every uu and vv in $\mathcal{M}(\mathcal{R},\nu).\mathcal{M}(\mathcal{R},\nu)$. Let $X(\mathcal{R},\nu)X(\mathcal{R},\nu)$ and $Y(\mathcal{R},\nu)\,Y(\mathcal{R},\nu)$ be rearrangement-invariant spaces. We write $X(\mathcal{R},\nu)\to Y(\mathcal{R},\nu)X(\mathcal{R},\nu)\to Y(\mathcal{R},\nu)$ to denote that $X(\mathcal{R},\nu)$ $X(\mathcal{R},\nu)$ is continuously embedded into $Y(\mathcal{R},\nu).Y(\mathcal{R},\nu)$. One has that $X(\mathcal{R},\nu)\subset Y(\mathcal{R},\nu)X(\mathcal{R},\nu)\subset Y(\mathcal{R},\nu)$ if and only if $X(\mathcal{R},\nu)\to Y(\mathcal{R},\nu)X(\mathcal{R},\nu)\to Y(\mathcal{R},\nu)$.

Note that the embedding $X(\mathcal{R},\nu)\to Y(\mathcal{R},\nu)X(\mathcal{R},\nu)\to Y(\mathcal{R},\nu)$ holds if and only if there exists a constant CC such that

$\|\sum_j g_j\|_{Y(0,1)}\le C\sum_j\|g_j\|_{X(0,1)}\|\sum_j g_j\|_{Y(0,1)}\le C\sum_j\|g_j\|_{X(0,1)}$

for every $g_j\in\mathcal{M}_+(0,1)g_j\in\mathcal{M}_+(0,1)$.

Moreover, for any rearrangement-invariant spaces $X(\mathcal{R},\nu)X(\mathcal{R},\nu)$ and $Y(\mathcal{R},\nu)Y(\mathcal{R},\nu)$,

$X(\mathcal{R},\nu)\to Y(\mathcal{R},\nu)X(\mathcal{R},\nu)\to Y(\mathcal{R},\nu)$ if and only if

$$Y'(\mathcal{R},\nu)\to X'(\mathcal{R},\nu), \qquad (2.5)$$

with the same embedding constants.Given any $\epsilon>\frac{-1}{2},\epsilon>\frac{-1}{2}$, the dilation operator $E_{(1+2\epsilon)},E_{(1+2\epsilon)}$, defined at $f_j\in\mathcal{M}(0,1)$ $f_i\in\mathcal{M}(0,1)$ by

$$(E_{(1+2\epsilon)}f_j)(1-2\epsilon)=\begin{cases} f_j\left(\frac{1-2\epsilon}{1+2\epsilon}\right) & \text{if } \frac{1}{2\epsilon}>1\ge -1\\ 0 & \text{if } -1<1\le\frac{1}{2\epsilon}\end{cases},$$

is bounded on any rearrangement-invariant space $X(0,1),X(0,1)$, with norm not exceeding $\max\left\{1,\frac{1}{1+2\epsilon}\right\}$. $\max\left\{1,\frac{1}{1+2\epsilon}\right\}$. Hardy's Lemma tells us that if $f_{j_1},f_{j_2}\in\mathcal{M}_+(0,1),f_{j_1},f_{j_2}\in\mathcal{M}_+(0,1)$, and

$$\int_0^{1+2\epsilon}\sum_j f_{j_1}(r)dr\le\sum_j\int_0^{1+2\epsilon}f_{j_2}(r)dr \quad \text{for } \frac{-1}{2}<\epsilon<0,$$

then $\int_0^1\sum_j f_{j_1}(r)g_j(r)dr\le\sum_j\int_0^1 f_{j_2}(r)g_j(r)dr$ $\int_0^1\sum_j f_{j_1}(r)g_j(r)dr\le\sum_j\int_0^1 f_{j_2}(r)g_j(r)dr$for every non-increasing function $g_j:(0,1)\to[0,\infty].g_j:(0,1)\to[0,\infty]$. A consequence of this result is the Hardy-Littlewood-Pólya principle which asserts that if the functions $u,v\in\mathcal{M}(0,1)u,v\in\mathcal{M}(0,1)$ satisfy

$$\int_0^{1+2\epsilon} u^*(r)dr \le \int_0^{1+2\epsilon} v^*(r)dr \quad \text{for} \quad \frac{-1}{2}<\epsilon<0,$$

then $\|u\|_{X(\mathcal{R},\nu)} \le \|v\|_{X(\mathcal{R},\nu)}\|u\|_{X(\mathcal{R},\nu)} \le \|v\|_{X(\mathcal{R},\nu)}$ for every rearrangement-invariant space $X(\mathcal{R},\nu)X(\mathcal{R},\nu)$.

Since $\nu(\mathcal{R})<\infty, \nu(\mathcal{R})<\infty$, for every rearrangement-invariant space $X(\mathcal{R},\nu)X(\mathcal{R},\nu)$one has that

$$L^\infty(\mathcal{R},\nu)\to X(\mathcal{R},\nu)\to L^1(\mathcal{R},\nu). \qquad (2.6)$$

Throughout, we use the convention that $\frac{1}{\infty}=0.\frac{1}{\infty}=0$.

A basic example of a function norm is the Lebesgue norm $\|\cdot\|_{L^{(1+\epsilon,1-\epsilon)}(0,1)},\|\cdot\|_{L^{(1+\epsilon,1-\epsilon)}(0,1)}$, defined as usual for $0\le\epsilon\le\infty$. $0\le\epsilon\le\infty$.Assume that $-1<\epsilon,\epsilon\ge-\infty.-1<\epsilon,\epsilon\ge-\infty$. We defined the functional $\|\cdot\|_{L^{(1+\epsilon,1-\epsilon)}(0,1)}\|\cdot\|_{L^{(1+\epsilon,1-\epsilon)}(0,1)}$ by

$$\left\|\sum_j f_j\right\|_{L^{(1+\epsilon,1+\epsilon)}(0,1)} = \left\|(1-2\epsilon)^{\left(\frac{1}{1+\epsilon}-\frac{1}{1-\epsilon}\right)}\sum_j f_j^*(1-2\epsilon)\right\|_{L^{(1+\epsilon)}(0,1)}$$

$$\left\|\sum_j f_j\right\|_{L^{(1+\epsilon,1+\epsilon)}(0,1)} = \left\|(1-2\epsilon)^{\left(\frac{1}{1+\epsilon}-\frac{1}{1-\epsilon}\right)}\sum_j f_j^*(1-2\epsilon)\right\|_{L^{(1+\epsilon)}(0,1)}$$

for $f_j\in\mathcal{M}_+(0,1).f_j\in\mathcal{M}_+(0,1)$.If either $0<\epsilon<\infty 0<\epsilon<\infty$ and $0\le-\epsilon\le\infty,0\le-\epsilon\le\infty$, or $\epsilon=0,\epsilon=0$, or $\epsilon=\infty$, $\epsilon=\infty$, then $\|\cdot\|_{L^{(1+\epsilon,1+\epsilon)}(0,1)}\|\cdot\|_{L^{(1+\epsilon,1+\epsilon)}(0,1)}$ is equivalent to a rearrangement-invariant function norm, and

$$\left(L^{(1+\epsilon,1-\epsilon)}\right)'(0,1)=L^{(1+\epsilon)',(1-\epsilon)'}(0,1). \qquad (2.7)$$

We further define the functional $\|\cdot\|_{L^{(1+\epsilon,1-\epsilon)}(0,1)}\|\cdot\|_{L^{(1+\epsilon,1-\epsilon)}(0,1)}$ as

$$\|\sum_j f_j\|_{L^{(1+\epsilon,1-\epsilon)}(0,1)} = \left\|(1-2\epsilon)^{\left(\frac{1}{1+\epsilon}-\frac{1}{1-\epsilon}\right)}\sum_j f_j^{**}(1-2\epsilon)\right\|_{L^{1-\epsilon}(0,1)}$$

$$\|\sum_j f_j\|_{L^{(1+\epsilon,1-\epsilon)}(0,1)} = \left\|(1-2\epsilon)^{\left(\frac{1}{1+\epsilon}-\frac{1}{1-\epsilon}\right)}\sum_j f_j^{**}(1-2\epsilon)\right\|_{L^{1-\epsilon}(0,1)}$$

for $f_j \in \mathcal{M}_+(0,1).f_j \in \mathcal{M}_+(0,1)$.

If either $-1<\epsilon<\infty-1<\epsilon<\infty$ and $0\geq\epsilon\geq\infty,0\geq\epsilon\geq\infty$, or $\epsilon=\infty,\epsilon=\infty$, „ then$\|\cdot\|_{L^{(1+\epsilon,1+\epsilon)}(0,1)}\|\cdot\|_{L^{(1+\epsilon,1+\epsilon)}(0,1)}$ is a rearrangement-invariant function norm (see e.g. [31] Theorem 9.7.5). The norm $\|\cdot\|_{L^{(1+\epsilon,1-\epsilon)}(0,1)}\|\cdot\|_{L^{(1+\epsilon,1-\epsilon)}(0,1)}$ and $\|\cdot\|_{L^{(1+\epsilon,1-\epsilon)}(0,1)}\|\cdot\|_{L^{(1+\epsilon,1-\epsilon)}(0,1)}$ are called Lorentz function norms, and the corresponding spaces $L^{(1+\epsilon,1-\epsilon)}(\mathcal{R},\nu)L^{(1+\epsilon,1-\epsilon)}(\mathcal{R},\nu)$ and $L^{(1+\epsilon,1-\epsilon)}(\mathcal{R},\nu)L^{(1+\epsilon,1-\epsilon)}(\mathcal{R},\nu)$ are called Lorentz spaces. Suppose now that $-1<\epsilon,\epsilon\geq\infty-1<\epsilon,\epsilon\geq\infty$ and $\alpha\in\mathbb{R}$. $\alpha\in\mathbb{R}$. We define the functional $\|\cdot\|_{L^{(1+\epsilon,1-\epsilon);\alpha}(0,1)}\|\cdot\|_{L^{(1+\epsilon,1-\epsilon);\alpha}(0,1)}$ by

$$\left\|\sum_j f_j\right\|_{L^{(1+\epsilon,1-\epsilon);\alpha}(0,1)} = \left\|(1-2\epsilon)^{\left(\frac{1}{1+\epsilon}-\frac{1}{1-\epsilon}\right)}\log^{\alpha}\left(\frac{e}{1-2\epsilon}\right)\sum_j f_j^{*}(1-2\epsilon)\right\|_{L^{(1-\epsilon)}(0,1)} \qquad (2.8)$$

for $f_j \in \mathcal{M}_+(0,1).f_j \in \mathcal{M}_+(0,1)$.

For suitable choices of $(1+\epsilon,1-\epsilon),(1-\epsilon),\alpha,\|\cdot\|_{L^{(1+\epsilon,1-\epsilon);\alpha}(0,1)}$ $(1+\epsilon,1-\epsilon),(1-\epsilon),\alpha,\|\cdot\|_{L^{(1+\epsilon,1-\epsilon);\alpha}(0,1)}$ is equivalent to a function norm. If this is the case, $\|\cdot\|_{L^{(1+\epsilon,1-\epsilon);\alpha}(0,1)}\|\cdot\|_{L^{(1+\epsilon,1-\epsilon);\alpha}(0,1)}$ is called a Lorentz-Zygmund function norm, and the corresponding space $L^{(1+\epsilon,1-\epsilon);\alpha}(\mathcal{R},\nu)L^{(1+\epsilon,1-\epsilon);\alpha}(\mathcal{R},\nu)$ is called a Lorentz-Zygmundspace. The space $L^{\infty,\frac{1}{1+2\epsilon};-\frac{m}{n},-1}(\lambda_d)L^{\infty,\frac{1}{1+2\epsilon};-\frac{m}{n},-1}(\lambda_d)$ mentioned in Example 5.6 is the so-called generalized Lorentz-Zygmund space corresponding to the function norm given by

$$\left\|\sum_j f_j\right\|_{L^{\infty,\frac{n}{m};-\frac{m}{n},-1}(0,1)} = \left\|(1-2\epsilon)^{-\frac{m}{n}}\log^{-\frac{m}{n}}\left(\frac{e}{1-2\epsilon}\right)\left(\log\left(1+\log\left(\frac{e}{1-2\epsilon}\right)\right)\right)^{-1}\sum_j f_j^*(1-2\epsilon)\right\|_{L^{\frac{n}{m}}(0,1)}$$

for $\sum_j f_j \in \mathcal{M}_+(0,1)\sum_j f_j \in \mathcal{M}_+(0,1)$ (see [29],[20],[31],Chapter 9).The following inclusion relations between Lorentz spaces hold:

$$L^{(1+\epsilon,1-\epsilon)}(0,1) = L^{1+\epsilon}(0,1) \qquad (2.9)$$

for $0 \le \epsilon \le \infty;0 \le \epsilon \le \infty;$

$$L^{(1+\epsilon,1-\epsilon)}(0,1) \to L^{(1+\epsilon,r)}(0,1) \qquad (2.10)$$

if $1 \le 1-\epsilon \le r \le \infty; 1 \le 1-\epsilon \le r \le \infty;$

$$L^{(1+\epsilon,1-\epsilon)}(0,1) \to L^{(1+\epsilon,1-\epsilon)}(0,1) \qquad (2.11)$$

for $0 \le \epsilon \le \infty, 0 \ge \epsilon \ge -\infty;0 \le \epsilon \le \infty, 0 \ge \epsilon \ge -\infty;$

if either $0 < \epsilon < \infty 0 < \epsilon < \infty$ and $1 \le \epsilon \le \infty, 1 \le \epsilon \le \infty$, or $\epsilon = \infty$, $\qquad (2.12) \epsilon = \infty, \qquad (2.12)$

then $L^{(1+\epsilon,1-\epsilon)}(0,1) = L^{(1+\epsilon,1-\epsilon)}(0,1)$ $L^{(1+\epsilon,1-\epsilon)}(0,1) = L^{(1+\epsilon,1-\epsilon)}(0,1)$ up to equivalent norms; $(L^{\infty,1-\epsilon;-1})'(0,1) \to L^{(1+\epsilon,(1-\epsilon)')}(0,1)$ (2.13)

for $\epsilon < 0, \epsilon < 0$, up to equivalent norms. For the last property, (see e.g. [33]).

A function $A: [0,\infty) \to [0,\infty] A: [0,\infty) \to [0,\infty]$ is called a Young function if it is convex (non trivial), left-continuous and vanishes at 00. Thus, any such function takes the form

$A(1-2\epsilon) \int_0^{1-2\epsilon} a(\tau)d\tau$ $A(1-2\epsilon) \int_0^{1-2\epsilon} a(\tau)d\tau$ for $\epsilon \geq \frac{1}{2}$, (2.14) $\epsilon \geq \frac{1}{2}$, (2.14)

for some non-decreasing, left-continuous function $a: [0,\infty) \to [0,\infty]. a: [0,\infty) \to [0,\infty]$. The Luxemburg function norm $\|\cdot\|_{L^A(0,1)} \|\cdot\|_{L^A(0,1)}$ is defined by

$$\left\|\sum_j f_j\right\|_{L^A(0,1)} = \inf\left\{\lambda > 0: \int_0^1 A\left(\frac{\sum_j f_j(1-2\epsilon)}{\lambda}\right) d(1-2\epsilon) \leq 1\right\}$$
$$\left\|\sum_j f_j\right\|_{L^A(0,1)} = \inf\left\{\lambda > 0: \int_0^1 A\left(\frac{\sum_j f_j(1-2\epsilon)}{\lambda}\right) d(1-2\epsilon) \leq 1\right\},$$

for $f_j \in \mathcal{M}_+(0,1). f_j \in \mathcal{M}_+(0,1)$.

The corresponding rearrangement-invariant space $L^A(\mathcal{R},\nu)$ $L^A(\mathcal{R},\nu)$ is called a Orlicz space. In particular, $L^A(0,1) = L^{1+\epsilon}(0,1)$ $L^A(0,1) = L^{1+\epsilon}(0,1)$ if $A(1-2\epsilon) = (1-2\epsilon)^{1+\epsilon}$ $A(1-2\epsilon) = (1-2\epsilon)^{1+\epsilon}$ for some $0 \leq \epsilon < \infty, 0 \leq \epsilon < \infty$, and $L^A(0,1) = L^\infty(0,1) L^A(0,1) = L^\infty(0,1)$ if $A(1-2\epsilon) = 0$ $A(1-2\epsilon) = 0$ for $\frac{1}{2} \leq \epsilon \leq 0, \frac{1}{2} \leq \epsilon \leq 0$, and $A(1-2\epsilon) = \infty$ $A(1-2\epsilon) = \infty$ for $\epsilon > 0. \epsilon > 0$.

Given two Young functions $A\,A$ and BB, the function norms $\|\cdot\|_{L^A(0,1)}\|\cdot\|_{L^A(0,1)}$ and $\|\cdot\|_{L^B(0,1)}\|\cdot\|_{L^B(0,1)}$ are equivalent if and only if AA and BB are equivalent near infinity, in the sense that there exist constants $c \geq 1c \geq 1$ and $(1-2\epsilon)_0 \geq 0(1-2\epsilon)_0 \geq 0$ such that

$$A\left(\frac{1-2\epsilon}{c}\right) \leq B(1-2\epsilon) \leq A\big(c(1-2\epsilon)\big)$$

$$A\left(\frac{1-2\epsilon}{c}\right) \leq B(1-2\epsilon) \leq A\big(c(1-2\epsilon)\big) \qquad \text{for}$$

$(1-2\epsilon) \geq (1-2\epsilon)_0.(1-2\epsilon) \geq (1-2\epsilon)_0.$

A common extension of Orlicz and Lorentz spaces is provided by a family of Orlicz-Lorentz spaces. Given $0 < \epsilon \leq \infty, 0 \geq \epsilon > -\infty$ $0 < \epsilon \leq \infty, 0 \geq \epsilon > -\infty$ and a Young function DD such that

$$\int^{\infty} \frac{D(1-2\epsilon)}{(1-2\epsilon)^{\epsilon+2}}\, d(1-2\epsilon) < \infty,$$

we denote by $\|\cdot\|_{L(1+\epsilon,1-\epsilon,D)(0,1)}\|\cdot\|_{L(1+\epsilon,1-\epsilon,D)(0,1)}$ the Orlicz-Lorentz rearrangement-invariant function norm defined as

$$\left\|\sum_j f_j\right\|_{L(1+\epsilon,1-\epsilon,D)(0,1)} = \left\|(1+2\epsilon)^{-\frac{1}{1+\epsilon}}\sum_j f_j^*\left((1+2\epsilon)^{\frac{1}{1-\epsilon}}\right)\right\|_{L^D(0,1)} \qquad (2.15)$$

for $f_j \in \mathcal{M}_+(0,1).f_j \in \mathcal{M}_+(0,1)$.The fact that (2.2.15) actually defines a function norm follows from simple variants in the proof of ([10], Proposition 2.1). Given a measure space $(\mathcal{R},\nu)(\mathcal{R},\nu)$,we denote by $L(1+\epsilon,1-\epsilon,D)(\mathcal{R},\nu)L(1+\epsilon,1-\epsilon,D)(\mathcal{R},\nu)$ the Orlicz-Lorentz space associated with the rearrangement-invariant function norm $\|\cdot\|_{L(1+\epsilon,1-\epsilon,D)(0,1)}.\|\cdot\|_{L(1+\epsilon,1-\epsilon,D)(0,1)}$. Note that this class of Orlicz-Lorentz spaces includes (up to equivalent norms) the Orlicz spaces and various instances of Lorentz and Lorentz-Zygmund spaces.

2.2 Sobolev spaces. An open set $\lambda\lambda$ in $\mathbb{R}^n\mathbb{R}^n$ is said to have the cone property if there exists a finite cone $\Lambda\Lambda$ such that each point in $\lambda\lambda$ is the vertex of finite cone contained in $\lambda\lambda$ and congruent to $\Lambda\Lambda$. An open set $\lambda\lambda$ is called a Lipschitz domain if it is bounded and each point of $\partial\lambda\partial\lambda$ has a neighborhood $\mathcal{U}\mathcal{U}$ such that $\lambda\cap\mathcal{U}\lambda\cap\mathcal{U}$ is the subgraph of a Lipschitz continuous function of $(n-1)(n-1)$ variables. Unless otherwise stated, in the remaining part of the section $\lambda\lambda$ will denote a bounded open set in $\mathbb{R}^n\mathbb{R}^n$ with the cone property. Let $m\in\mathbb{N}m\in\mathbb{N}$ and let $X(\lambda)$ $X(\lambda)$ be a rearrangement-invariant space.We define the mm-th order Sobolev type space $W^mX(\lambda)W^mX(\lambda)$ as $W^mX(\lambda)=\{u\colon u$ $W^mX(\lambda)=\{u\colon u$ is mm-times weakly differentiable in λ,λ, and $|\nabla^k u|\in X(\lambda)|\nabla^k u|\in X(\lambda)$ for $k=0,\dots,m\},k=0,\dots,m\}$, equipped with the norm $\|u\|_{W^mX(\lambda)}=\sum_{k=0}^{m}\|\nabla^k u\|_{X(\lambda)}$. $\|u\|_{W^mX(\lambda)}=\sum_{k=0}^{m}\|\nabla^k u\|_{X(\lambda)}$. Here, $\nabla^m u\nabla^m u$ denotes the vector of all mm-th order weak derivatives of uu. In particular, $\nabla^0 u\nabla^0 u$ stands for uu, and $\nabla^1 u\nabla^1 u$ will also be simply denoted by $\nabla u\nabla u$. The subspace $W_\perp^mX(\lambda)W_\perp^mX(\lambda)$ of $W^mX(\lambda)W^mX(\lambda)$ is defined as

$$W_\perp^mX(\lambda)=\left\{u\in W^mX(\lambda)\colon \int_\lambda \nabla^k u\,dx=0,\ \text{ for } 0\le k\le m-1\right\}.$$

The notation $V^mX(\lambda)V^mX(\lambda)$ will be employed to denote the space

$V^mX(\lambda)=\{u\colon uV^mX(\lambda)=\{u\colon u$ is mm-times weakly differentiable in λ,λ, and $|\nabla^k u| \in X(\lambda)\}$, $|\nabla^k u| \in X(\lambda)\}$, equipped with the norm $\|u\|_{V^m X(\lambda)} = \sum_{k=0}^{m-1} \|\nabla^k u\|_{L^1(\lambda)} + \|\nabla^m u\|_{X(\lambda)}$. $\|u\|_{V^m X(\lambda)} = \sum_{k=0}^{m-1} \|\nabla^k u\|_{L^1(\lambda)} + \|\nabla^m u\|_{X(\lambda)}$.

Notethat,if$u \in V^m X(\lambda)$,$u \in V^m X(\lambda)$,then$|\nabla^m u| \in X(\lambda) \subset L^1(\lambda)$, $|\nabla^m u| \in X(\lambda) \subset L^1(\lambda)$, by property (P5) of rearrangement-invariant spaces. Hence, one actually has that $|\nabla^k u| \in L^1(\lambda)$ $|\nabla^k u| \in L^1(\lambda)$ for every $k = 0, \dots, m-1$,$k = 0, \dots, m-1$, by a standard Sobolev embedding on open sets with the cone property. The subspace $W_\perp^m X(\lambda)$ $W_\perp^m X(\lambda)$ of $V^m X(\lambda) V^m X(\lambda)$ is defined analogously to $W_\perp^m X(\lambda) W_\perp^m X(\lambda)$.

The spaces $W^m X(\lambda) W^m X(\lambda)$ and $V^m X(\lambda) V^m X(\lambda)$ are easily verified to be Banach spaces. In fact, they agree, up to equivalent norms.

Proposition 2.1. Let $\lambda\lambda$ be a bounded open set with the cone property in $\mathbb{R}^n, n \geq 2$,$\mathbb{R}^n, n \geq 2$, let $m \in \mathbb{N}$,$m \in \mathbb{N}$, and let $\|\cdot\|_{X(0,1)}$ $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Then $W^m X(\lambda) = V^m(\lambda)$.$W^m X(\lambda) = V^m(\lambda)$. Hence, $W_\perp^m X(\lambda) = V_\perp^m X(\lambda) W_\perp^m X(\lambda) = V_\perp^m X(\lambda)$ as well.

Proposition 2.1 is a straightforward consequence of ([17], Proposition 4.5) and of the fact that the isoperimetric function of any bounded connected open set with the cone property behaves like $(1+2\epsilon)^{1-\frac{1}{n}}(1+2\epsilon)^{1-\frac{1}{n}}$ as $(1+2\epsilon) \to 0^+ (1+2\epsilon) \to 0^+$ ([26], Corollary 5.2.1/3).

The next result deals with the equivalence of Sobolev and Poincaré trace inequalities.

Proposition 2.2. Let $\lambda\lambda$ be a bounded connected open set with the cone property in $\mathbb{R}^n, n \geq 2.\mathbb{R}^n, n \geq 2$. Let $m \in \mathbb{N} m \in \mathbb{N}$ and $d \in \mathbb{N}\, d \in \mathbb{N}$ be such that $1 \leq d \leq n\, 1 \leq d \leq n$ and $d \geq n - m$. $d \geq n - m$. Let $\|\cdot\|_{X(0,1)}\, \|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}\, \|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms.

Then the Sobolev trace embedding

$$Tr: W^m X(\lambda) \to Y(\lambda_d) \qquad (2.16)$$

holds if and only if Poincaré trace inequality

$$\|Tr\, u\|_{Y(\lambda_d)} \leq C \|\nabla^m u\|_{X(\lambda)} \qquad (2.17)$$

holds for some constant CC, and every $u \in W_{\perp}^m X(\lambda). u \in W_{\perp}^m X(\lambda)$. An ingredient in the proof of Proposition 2.2 is a (first-order) Poincaré type inequality which ensures that, if $\lambda\lambda$ is a bounded connected open set with the cone property, then there exists a constant $C = C(\lambda) C = C(\lambda)$ such that

$$\|u\|_{X(\lambda)} \leq C \|\nabla u\|_{X(\lambda)} \qquad (2.18)$$

for any r.i. function norm $\|\cdot\|_{X(\lambda)} \|\cdot\|_{X(\lambda)}$ and for every $u \in W_{\perp}^m X(\lambda)$. $u \in W_{\perp}^m X(\lambda)$. Inequality (2.182.18) is established in ([8], Lemma 4.1) in the special case when $\lambda\lambda$ is a ball. It proof makes use of a rearrangement type inequality for the norm $\|\nabla u\|_{X(\lambda)} \|\nabla u\|_{X(\lambda)}$ which holds, in fact, for Sobolev functions uu (with unrestricted boundary values) on any bounded connected open set $\lambda\lambda$ with the cone property ([15], Lemma 4.1 and inequality (3.5)).The proof in the general case is completely analogous, provided that balls are replaced with an arbitrary bounded connected open set with the cone property ([19].

Proof. Assume that (2.12.1) is in force.Given $k \in \mathbb{N}, k \in \mathbb{N}$, denote by $\mathcal{P}^k \mathcal{P}^k$ the space of polynomials whose degree does not exceed k

k. Given any $u \in W^m X(\lambda), u \in W^m X(\lambda)$, there exists a (unique) polynomial $P_u \in \mathcal{P}^{m-1} P_u \in \mathcal{P}^{m-1}$ such that $u - P_u \in W_\perp^m X(\lambda)$. $u - P_u \in W_\perp^m X(\lambda)$. Moreover, the coefficients of $P_u P_u$ are linear combinations of the components of

$\int_{\mathbb{R}^n} \nabla^k u dx, \int_{\mathbb{R}^n} \nabla^k u dx$, for $k = 0, \dots, m-1, k = 0, \dots, m-1$, with coefficients depending on nn and mm. Thus, given any $u \in W^m X(\lambda), u \in W^m X(\lambda)$,

$$\|Tr\, u\|_{Y(\lambda_d)} \le \|Tr\, u - P_u\|_{Y(\lambda_d)} + \|P_u\|_{Y(\lambda_d)} = \|Tr\,(u - P_u)\|_{Y(\lambda_d)} + \|P_u\|_{Y(\lambda_d)}$$

$$\le C\|\nabla^m u\|_{X(\lambda)}$$

$$+ \sum_{k=0}^{m-1} C' \int_\lambda |\nabla^k u| dx \sum_{h_i \in \mathbb{N}\cup\{0\} \sum_{i=1}^n h_i = k} \left\| |x_1|^{h_1} \dots |x_n|^{h_n} \right\|_{Y(\lambda_d)}$$

$$\le C\|\nabla^m u\|_{X(\lambda)} + \sum_{k=0}^{m-1} C'' \int_\lambda |\nabla^k u| dx$$

$$\le C\|\nabla^m u\|_{X(\lambda)} + \sum_{k=0}^{m-1} C'' \|1\|_{X'(\lambda)} \left\|\nabla^k u\right\|_{X(\lambda)} \le C''' \|u\|_{W^m X(\lambda)},$$

where $C\, C$ is the constant appearing in (2.2.17), and C', C'' C', C'' and $C''' C'''$ are suitable constants depending on n, mn, m, $\lambda_d \|\cdot\|_{Y(0,1)}, C.\ \lambda_d \|\cdot\|_{Y(0,1)}, C.$ Embedding (2.16) is thus established. Conversely, assume that (2.16) holds. An iterated use of the Poincaré inequality (2.18) tells us that there exists a constant $C = C(m, \lambda) C = C(m, \lambda)$ such that

$$\|u\|_{X(\lambda)} \le C\|\nabla u\|_{X(\lambda)} \le C^2\|\nabla^2 u\|_{X(\lambda)} \le \dots \le C^m \|\nabla^m u\|_{X(\lambda)} \qquad (2.19)$$

for $u \in W_\perp^m X(\lambda) u \in W_\perp^m X(\lambda)$. Inequality (2.17) follows from (2.16) and (2.19).

3. Preliminary technical results we say that an operator

$T: \mathcal{M}_+(0,1) \to \mathcal{M}_+(0,1) T: \mathcal{M}_+(0,1) \to \mathcal{M}_+(0,1)$

is bounded between two rearrangement-invariant spaces $X(0,1)$ $X(0,1)$ and $Y(0,1), Y(0,1)$, and we write

$$T: X(0,1) \to Y(0,1), \qquad (3.1)$$

if the quantity

$$\|T\| = \sup\left\{\|T\sum_j f_j\|_{Y(0,1)} : f_j \in X(0,1) \cap \mathcal{M}_+(0,1), \|\sum_j f_j\|_{X(0,1)} \le 1\right\}$$

$$\|T\| = \sup\left\{\|T\sum_j f_j\|_{Y(0,1)} : f_j \in X(0,1) \cap \mathcal{M}_+(0,1), \|\sum_j f_j\|_{X(0,1)} \le 1\right\}$$

is finite. Such a quantity will be called the norm of TT. The space $Y(0,1)Y(0,1)$will be called optimal, within a certain class, in (3.1) if, whenever $Z(0,1)Z(0,1)$ is another rearrangement-invariant space, from the same class, such that $T: X(0,1) \to Z(0,1)$ $T: X(0,1) \to Z(0,1)$, we have that $Y(0,1) \to Z(0,1)$. $Y(0,1) \to Z(0,1)$. Equivalently, the corresponding function norm $\|\cdot\|_{Y(0,1)}\|\cdot\|_{Y(0,1)}$ will be said to be optimal in (3.1) in the relevant class.Assume that $T, T': \mathcal{M}_+(0,1) \to \mathcal{M}_+(0,1)$ $T, T': \mathcal{M}_+(0,1) \to \mathcal{M}_+(0,1)$

are operators such that

$$\int_0^1 T\sum_j f_j(1+2\epsilon) g_j(1+2\epsilon) d(1+2\epsilon) = \int_0^1 \sum_j f_j(1+2\epsilon) T' g_j(1+2\epsilon) d(1+2\epsilon)$$

$$\int_0^1 T\sum_j f_j(1+2\epsilon) g_j(1+2\epsilon) d(1+2\epsilon) = \int_0^1 \sum_j f_j(1+2\epsilon) T' g_j(1+2\epsilon) d(1+2\epsilon)$$

for every $f_j, g_j \in \mathcal{M}_+(0,1) f_j, g_j \in \mathcal{M}_+(0,1)$.

Let $X(0,1)X(0,1)$ and $Y(0,1)Y(0,1)$ be rearrangement-invariant spaces. A simple argument involving Fubini's theorem and the definition of the associate norm shows that

$T: X(0,1) \to Y(0,1) T: X(0,1) \to Y(0,1)$ if and

only if $T': X'(0,1) \to Y'(0,1)$ (3.2)
$T': X'(0,1) \to Y'(0,1)$ (3.2)
and $\|T\| = \|T'\|\|T\| = \|T'\|$ (see e.g. [17], Lemma 8.1).

Let $\omega: (0,1) \to (0,\infty)\omega: (0,1) \to (0,\infty)$ be a measurable function, and let $0 > \epsilon > -\infty 0 > \epsilon > -\infty$ be such that

$$\left\|\omega(1-2\epsilon)(1-2\epsilon)^{(1-\epsilon)}\right\|_{L^\infty(0,1)} < \infty. \qquad (3.3)$$

$$\left\|\omega(1-2\epsilon)(1-2\epsilon)^{(1-\epsilon)}\right\|_{L^\infty(0,1)} < \infty. \qquad (3.3)$$

We define the operators $J_{(\omega,1-\epsilon)}J_{(\omega,1-\epsilon)}$ and $J'_{(\omega,1-\epsilon)}J'_{(\omega,1-\epsilon)}$ at every $g_j \in \mathcal{M}_+(0,1)g_j \in \mathcal{M}_+(0,1)$ by

$$J_{(\omega,1-\epsilon)}g_j(1-2\epsilon) = \int_{(1-2\epsilon)^{\frac{1}{1-\epsilon}}}^{1} \omega(1+2\epsilon)\sum_j g_j(1+2\epsilon)d(1+2\epsilon),$$

$$J'_{(\omega,1-\epsilon)}g_j(1-2\epsilon) = \omega(1-2\epsilon)(3.4)$$

For $\frac{1}{2} < \epsilon < 0.\frac{1}{2} < \epsilon < 0.$ Assume that $\|\cdot\|_{X(0,1)}\|\cdot\|_{X(0,1)}$ is a rearrangement-invariant function norm.

We then define the functional $\|\cdot\|_{X'_{(\omega,1-\epsilon)}(0,1)}\|\cdot\|_{X'_{(\omega,1-\epsilon)}(0,1)}$ by

$$\left\|\sum_j g_j\right\|_{X'_{(\omega,1-\epsilon)}(0,1)} = \left\|J'_{(\omega,1-\epsilon)}\sum_j g_j^*\right\|_{X'(0,1)} \qquad (3.5)$$

for $g_j \in \mathcal{M}_+(0,1)g_j \in \mathcal{M}_+(0,1)$, where $\|\cdot\|_{X'(0,1)}\|\cdot\|_{X'(0,1)}$ is the associate norm $\|\cdot\|_{X(0,1)}.\|\cdot\|_{X(0,1)}.$

Proposition 3.1. Let $\omega: (0,1) \to (0,\infty)\omega: (0,1) \to (0,\infty)$ be a measurable function, and let $1 > \epsilon > -\infty 1 > \epsilon > -\infty$. Assume that (3.3) holds. Let $\|\cdot\|_{X(0,1)}$ $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Then the functional $\|\cdot\|_{X'_{(\omega,1-\epsilon)}(0,1)}$, $\|\cdot\|_{X'_{(\omega,1-\epsilon)}(0,1)}$, defined by (3.5), is a rearrangement-invariant function norm. Moreover, on denoting by $\|\cdot\|_{X_{(\omega,1-\epsilon)}(0,1)}\|\cdot\|_{X_{(\omega,1-\epsilon)}(0,1)}$ the rearrangement-invariant function norm whose associate norm is $\|\cdot\|_{X'_{(\omega,1-\epsilon)}(0,1)}$,$\|\cdot\|_{X'_{(\omega,1-\epsilon)}(0,1)}$, one has that $J_{(\omega,1-\epsilon)}: X(0,1) \to X_{(\omega,1-\epsilon)}(0,1) \quad (3.6)$

$$J_{(\omega,1-\epsilon)}: X(0,1) \to X_{(\omega,1-\epsilon)}(0,1) \quad (3.6)$$

with norm not exceeding 1, namely,

$$\left\| \int_{(1-2\epsilon)^{\frac{1}{1-\epsilon}}}^{1} \omega(1+2\epsilon) \sum_j g_j(1+2\epsilon)\,d(1+2\epsilon) \right\|_{X_{(\omega,1-\epsilon)}(0,1)} \leq \sum_j \|g_j\|_{X(0,1)} \quad (3.7)$$

for $g_j \in X(0,1)$.$g_j \in X(0,1)$.The function norm $\|\cdot\|_{X_{(\omega,1-\epsilon)}(0,1)}$ $\|\cdot\|_{X_{(\omega,1-\epsilon)}(0,1)}$is optimal in (3.6) among all rearrangement-invariant function norms.

Proof. We begin by showing that the functional$\|\cdot\|_{X'_{(\omega,1-\epsilon)}(0,1)}$ $\|\cdot\|_{X'_{(\omega,1-\epsilon)}(0,1)}$ satisfies axioms (P1)-(P6) of the definition of rearrangement-invariant function norm. Let $f_j, g_j \in \mathcal{M}_+(0,1)$. $f_j, g_j \in \mathcal{M}_+(0,1)$. Then, by (2.1),

$$\int_0^{(1-2\epsilon)^{1-\epsilon}} \sum_j (f_j+g_j)^*(1+2\epsilon)\,d(1+2\epsilon) \leq (1+2\epsilon)$$

$$+\int_0^{(1-2\epsilon)^{1-\epsilon}} \sum_j g_j^*(1+2\epsilon)\,d(1+2\epsilon)$$

$+\int_0^{(1-2\epsilon)^{1-\epsilon}} \sum_j g_j^*(1+2\epsilon)\,d(1+2\epsilon)$ for $\frac{1}{2} < \epsilon < 0.\frac{1}{2} < \epsilon < 0.$
Hence, owning to the Hardy-Littlewood-Pólya principle,

$$\left\|\sum_j f_j + g_j\right\|_{X'_{(\omega,1-\epsilon)}(0,1)} \leq \sum_j \|f_j\|_{X'_{(\omega,1-\epsilon)}(0,1)} + \sum_j \|g_j\|_{X'_{(\omega,1-\epsilon)}(0,1)}.$$

This proves the triangle inequality. The remaining properties in axiom (P1), as well as axioms (P2),(P3) and (P6), are trivially satisfied. By (3.3) and (2.6), there exists a positive constant CC such that

$$\|\chi_{(0,1)}\|_{X'_{(\omega,1-\epsilon)}(0,1)} = \|\omega(1-2\epsilon)(1-2\epsilon)^{1-\epsilon}\|_{X'(0,1)}$$
$$\leq C\|\omega(1-2\epsilon)(1-2\epsilon)^{1-\epsilon}\|_{L^\infty(0,1)} < \infty,$$

whence (P4) holds. To verify (P5), first note that there exists a positive constant cc, depending only on $(1-\epsilon)(1-\epsilon)$, such that

$$\int_0^{2^{-(1-\epsilon)}} \sum_j g_j^*(1+2\epsilon)d(1+2\epsilon) \geq c\int_0^1 \sum_j g_j^*(1+2\epsilon)d(1+2\epsilon)$$
$$\int_0^{2^{-(1-\epsilon)}} \sum_j g_j^*(1+2\epsilon)d(1+2\epsilon) \geq c\int_0^1 \sum_j g_j^*(1+2\epsilon)d(1+2\epsilon)$$

for every

$g_j \in \mathcal{M}_+(0,1) g_j \in \mathcal{M}_+(0,1)$. Thus, by (P5) for the function norm $\|\cdot\|_{X'(0,1)}, \|\cdot\|_{X'(0,1)}$, there exists a positive constant $c'c'$ such that

$$\left\|\omega(1-2\epsilon)\int_0^{(1-2\epsilon)^{1-\epsilon}} \sum_j g_j^*(1+2\epsilon)d(1+2\epsilon)\right\|_{X'(0,1)}$$
$$\geq c'\int_0^1 \omega(1-2\epsilon)\int_0^{(1-2\epsilon)^{1-\epsilon}} \sum_j g_j^*(1+2\epsilon)d(1+2\epsilon)\, d(1-2\epsilon)$$
$$\geq c'\int_{\frac{1}{2}}^1 \omega(1-2\epsilon)d(1-2\epsilon)\int_0^{2^{-(1-\epsilon)}} \sum_j g_j^*(1+2\epsilon)d(1+2\epsilon)$$
$$= c''\int_0^1 \sum_j g_j^*(1+2\epsilon)d(1+2\epsilon)$$

where $c'' = c' \int_{\frac{1}{2}}^{1} \omega(1-2\epsilon)d(1-2\epsilon)$, $c'' = c' \int_{\frac{1}{2}}^{1} \omega(1-2\epsilon)d(1-2\epsilon)$, whence (P5) follows [19]. We shall now show (3.7). By (3.2) and (3.4), this is equivalent to establishing that $J'_{(\omega,1-\epsilon)} J'_{(\omega,1-\epsilon)}$: $X'_{(\omega,1-\epsilon)}(0,1) \to X'(0,1)$ $X'_{(\omega,1-\epsilon)}(0,1) \to X'(0,1)$ with constant not exceeding one, that is,

$$\left\| \omega(1-2\epsilon) \int_0^{(1-2\epsilon)^{1-\epsilon}} \sum_j g_j(1+2\epsilon)d(1+2\epsilon) \right\|_{X'(0,1)} \leq \sum_j \|g_j\|_{X'_{(\omega,1-\epsilon)}(0,1)} \quad (3.8)$$

for $g_j \in \mathcal{M}_+(0,1)$ $g_j \in \mathcal{M}_+(0,1)$. By the very definition of the norm $\|\cdot\|_{X'_{(\omega,1-\epsilon)}(0,1)}$ $\|\cdot\|_{X'_{(\omega,1-\epsilon)}(0,1)}$,

$$\left\| \omega(1-2\epsilon) \int_0^{(1-2\epsilon)^{1-\epsilon}} \sum_j g_j^*(1+2\epsilon)d(1+2\epsilon) \right\|_{X'(0,1)} = \left\| \sum_j g_j \right\|_{X'_{(\omega,1-\epsilon)}(0,1)}$$

for $g_j \in \mathcal{M}_+(0,1)$ $g_j \in \mathcal{M}_+(0,1)$. Moreover, by the Hardy-Littlewood inequality (2.2) and property (P2) of the norm $\|\cdot\|_{X'(0,1)}$, $\|\cdot\|_{X'(0,1)}$,

$$\left\| \omega(1-2\epsilon) \int_0^{(1-2\epsilon)^{1-\epsilon}} \sum_j g_j(1+2\epsilon)d(1+2\epsilon) \right\|_{X'(0,1)}$$

$$\leq \left\| \omega(1-2\epsilon) \int_0^{(1-2\epsilon)^{1-\epsilon}} \sum_j g_j^*(1+2\epsilon)d(1+2\epsilon) \right\|_{X'(0,1)}$$

for $g_j \in \mathcal{M}_+(0,1)$ $g_j \in \mathcal{M}_+(0,1)$. Hence, (3.8) follows.

It remains to prove that $\|\cdot\|_{X_{(\omega,1-\epsilon)}(0,1)}$ $\|\cdot\|_{X_{(\omega,1-\epsilon)}(0,1)}$ is the optimal rearrangement-invariant function norm that renders (3.6) true. In order to verify this fact, assume that $\|\cdot\|_{Y(0,1)}$ $\|\cdot\|_{Y(0,1)}$ is another rearrangement-invariant function norm such that $J_{(\omega,1-\epsilon)}: (0,1)X(0,1) \to Y(0,1)$. $J_{(\omega,1-\epsilon)}: (0,1)X(0,1) \to Y(0,1)$. Then, by (3.2), $J'_{(\omega,1-\epsilon)}: Y'(0,1) \to X'(0,1) J'_{(\omega,1-\epsilon)}: Y'(0,1) \to X'(0,1)$, namely $\left\|J'_{(\omega,1-\epsilon)} \sum_j g_j\right\|_{X'(0,1)} \le C \sum_j \|g_j\|_{Y'(0,1)}$ $\left\|J'_{(\omega,1-\epsilon)} \sum_j g_j\right\|_{X'(0,1)} \le C \sum_j \|g_j\|_{Y'(0,1)}$ for some positive constant CC and every $g_j \in \mathcal{M}_+(0,1)$ $g_j \in \mathcal{M}_+(0,1)$. Hence, by the rearrangement-invariant of the space

$Y'(0,1), \left\|J'_{(\omega,1-\epsilon)} \sum_j g_j^*\right\|_{X'(0,1)} \le C \sum_j \|g_j^*\|_{Y'(0,1)} = C\left\|\sum_j g_j\right\|_{Y'(0,1)}$ $Y'(0,1), \left\|J'_{(\omega,1-\epsilon)} \sum_j g_j^*\right\|_{X'(0,1)} \le C \sum_j \|g_j^*\|_{Y'(0,1)} = C\left\|\sum_j g_j\right\|_{Y'(0,1)}$ for every $g_j \in \mathcal{M}_+(0,1)$ $g_j \in \mathcal{M}_+(0,1)$. Coupling this inequality with (3.5) shows that

$\left\|\sum_j g_j\right\|_{X'_{(\omega,1-\epsilon)}(0,1)} \le C \sum_j \|g_j\|_{Y'(0,1)}$

$\left\|\sum_j g_j\right\|_{X'_{(\omega,1-\epsilon)}(0,1)} \le C \sum_j \|g_j\|_{Y'(0,1)}$ for some positive constant CC and every $g_j \in \mathcal{M}_+(0,1), g_j \in \mathcal{M}_+(0,1)$, namely, $Y'(0,1) \to X'_{(\omega,1-\epsilon)}(0,1)$. $Y'(0,1) \to X'_{(\omega,1-\epsilon)}(0,1)$. Thus, by (2.5), $X_{(\omega,1-\epsilon)}(0,1) \to Y(0,1)$. $X_{(\omega,1-\epsilon)}(0,1) \to Y(0,1)$. This proves that $\|\cdot\|_{X_{(\omega,1-\epsilon)}(0,1)}$ $\|\cdot\|_{X_{(\omega,1-\epsilon)}(0,1)}$ is the optimal rearrangement-invariant function norm such that (3.6) holds. The proof is complete.The next two lemmas contain auxiliary results of technical nature.

Lemma 3.2. Let $\varrho \in \mathcal{M}_+(0,1)\varrho \in \mathcal{M}_+(0,1)$. Suppose that there exists a positive constant CC such

$$\varrho(1-2\epsilon) \leq \frac{C}{1-2\epsilon}\int_0^{1-2\epsilon} \varrho(1+2\epsilon)\,d(1+2\epsilon) \qquad (3.9)$$

for $\frac{1}{2} < \epsilon < 0.\frac{1}{2} < \epsilon < 0.$

Then, $\sup_{1-2\epsilon\leq 1+2\epsilon\leq 1} g_j^*(1+2\epsilon)\varrho(1+2\epsilon) \leq C(g_j^*\varrho)^{**}(1-2\epsilon)$ $\sup_{1-2\epsilon\leq 1+2\epsilon\leq 1} g_j^*(1+2\epsilon)\varrho(1+2\epsilon) \leq C(g_j^*\varrho)^{**}(1-2\epsilon)$ for $\frac{1}{2} < \epsilon < 0,\frac{1}{2} < \epsilon < 0,$ for every $g_j \in \mathcal{M}(0,1)g_j \in \mathcal{M}(0,1)$.

Proof. By (3.9) and the monotonicity of $g_j^*,g_j^*,$

$$\sup_{1-2\epsilon\leq 1+2\epsilon\leq 1} g_j^*(1+2\epsilon)\varrho(1+2\epsilon) \leq C \sup_{1-2\epsilon\leq 1+2\epsilon\leq 1} \frac{g_j^*(1+2\epsilon)}{1+2\epsilon}\int_0^{1+2\epsilon} \varrho(r)dr$$

$$\leq C \sup_{1-2\epsilon\leq 1+2\epsilon\leq 1} \frac{1}{1+2\epsilon}\int_0^{1+2\epsilon} \sum_j g_j^*(r)\varrho(r)\,dr$$

for $\frac{1}{2} < \epsilon < 0.\frac{1}{2} < \epsilon < 0.$ Owing to the Hardy-Littlewood inequality (2.2),

$\int_0^{1+2\epsilon} \sum_j g_j^*(r)\varrho(r)dr \leq \sum_j \int_0^{1+2\epsilon} \sum_j (g_j^*\varrho)^*(r)dr$
$\int_0^{1+2\epsilon} \sum_j g_j^*(r)\varrho(r)dr \leq \sum_j \int_0^{1+2\epsilon} \sum_j (g_j^*\varrho)^*(r)dr$ for $\frac{-1}{2} < \epsilon < 0.\frac{-1}{2} < \epsilon < 0.$ Hence,

$$\sup_{1-2\epsilon\le 1+2\epsilon\le 1} g_j^*(1+2\epsilon)\varrho(1+2\epsilon) \le C \sup_{1-2\epsilon\le 1+2\epsilon\le 1} \frac{1}{1+2\epsilon}\int_0^{1+2\epsilon}\sum_j (g_j^*\varrho)^*(r)dr$$
$$= \frac{C}{1-2\epsilon}\int_0^{1-2\epsilon}\sum_j (g_j^*\varrho)^*(r)dr = C(g_j^*\varrho)^{**}(1-2\epsilon)$$

for $\frac{1}{2} < \epsilon < 0.\frac{1}{2} < \epsilon < 0.$

Lemma 3.3 (i) Let $\theta \ge -1.\theta \ge -1.$ Then there exists a constant $C = C(\theta)C = C(\theta)$ such that

$$\int_0^{1-2\epsilon} \sup_{1+2\epsilon\le r\le 1-2\epsilon}\left(r^\theta \sum_j g_j^*(r)\right)d(1+2\epsilon)$$
$$\le C\int_0^{1-2\epsilon}(1+2\epsilon)^\theta \sum_j g_j^*(1+2\epsilon)d(1+2\epsilon) \qquad (3.10)$$

for every $g_j \in \mathcal{M}_+(0,1)g_j \in \mathcal{M}_+(0,1)$ and every $\frac{1}{2} < \epsilon < 0.$ $\frac{1}{2} < \epsilon < 0.$

(ii) Let $0 < \theta < 1.0 < \theta < 1.$ Then there exists a constant $C = C(\theta)C = C(\theta)$ such that

$$\int_0^{1-2\epsilon}(1+2\epsilon)^{-\theta} \sup_{1+2\epsilon\le r\le 1-2\epsilon}\left(r^\theta g_j^*(r)\right)d(1+2\epsilon)$$
$$\le C\int_0^{1-2\epsilon} g_j^*(1+2\epsilon)d(1+2\epsilon) \qquad (3.11)$$

for every $g_j \in \mathcal{M}_+(0,1)g_j \in \mathcal{M}_+(0,1)$ and every $\frac{1}{2} < \epsilon < 0.$ $\frac{1}{2} < \epsilon < 0.$

Proof. (i) Fix $\frac{1}{2} < \epsilon < 0. \frac{1}{2} < \epsilon < 0.$ Inequality (3.10) can be rewritten as

$$\int_0^\infty \chi_{(0,1-2\epsilon)}(1+2\epsilon) \sup_{1+2\epsilon\leq r<\infty} \left(r^\theta \chi_{(0,1-2\epsilon)}(r) g_j^*(r)\right) d(1+2\epsilon)$$
$$\leq C \int_0^\infty (1+2\epsilon)^\theta \chi_{(0,1-2\epsilon)}(1+2\epsilon) g_j^*(1+2\epsilon) d(1+2\epsilon). \qquad (3.12)$$

By ([23],Theorem 3.2), applied with

$$\epsilon = 0, \quad u(r) = r^\theta \chi_{(0,1-2\epsilon)}(r), \quad \omega(1+2\epsilon) = \chi_{(0,1-2\epsilon)},$$
$$v(1+2\epsilon) = (1+2\epsilon)^\theta \chi_{(0,1-2\epsilon)}(1+2\epsilon),$$

a necessary and sufficient condition for (3.12) is the validity of the inequality

$$\int_0^\tau \Big(\sup_{1+2\epsilon\leq r\leq\tau} u(r)\Big)\omega(1+2\epsilon) d(1+2\epsilon) \leq C' \int_0^\tau v(1+2\epsilon) d(1+2\epsilon) \quad (3.13)$$

for $0 < \tau < 1, 0 < \tau < 1,$

for some positive constant $C'.C'$. A close inspection of the proof of ([23], Theorem 3.2) reveals that CC is just an absolute constant multiple of $C'.C'$. It is easily verified that (3.13) holds with $C'C'$ depending only on θ,θ, and hence CC as well.

(ii) The same argument as in the case (i) can be used, but now with

$$\epsilon = 0, \quad u(r) = r^\theta \chi_{(0,1-2\epsilon)}(r), \quad \omega(1+2\epsilon) = \chi_{(0,1-2\epsilon)}(1+2\epsilon)(1+2\epsilon)^\theta,$$
$$v(1+2\epsilon) = \chi_{(0,1-2\epsilon)}(1+2\epsilon).$$

We now state and prove two key one-dimensional inequalities to be used in the proofs of our main results.

Theorem 3.4. Assume that $\alpha,\beta;\gamma,\delta \in (0,\infty)\alpha,\beta;\gamma,\delta \in (0,\infty)$ are such that

$\gamma+\delta \geq 1, \quad \alpha+\beta \geq 1$ $\gamma+\delta \geq 1, \quad \alpha+\beta \geq 1$ and $\alpha+\beta\gamma < 1. \qquad (3.14)$ $\alpha+\beta\gamma < 1. \qquad (3.14)$

Then there exists a positive constant $C = C(\alpha,\beta,\gamma,\delta)$, $C = C(\alpha,\beta,\gamma,\delta)$, such that, for every rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, $\|\cdot\|_{X(0,1)}$,

$$\left\| (1-2\epsilon)^{\alpha-1} \int_0^{(1-2\epsilon)^{\beta}} \left[\tau^{\gamma-1} \int_0^{\tau^{\delta}} \sum_j g_j^*(r)dr \right]^* (1+2\epsilon)d(1+2\epsilon) \right\|_{X(0,1)}$$

$$\leq C \left\| (1-2\epsilon)^{\alpha+\beta\gamma-1} \int_0^{\tau^{\beta\delta}} \sum_j g_j^*(r)dr \right\|_{X(0,1)} \qquad (3.15)$$

for $g_j \in \mathcal{M}_+(0,1)$ $g_j \in \mathcal{M}_+(0,1)$.

Proof. We begin by defining the operators J J and J' J' by

$$Jg_j(1-2\epsilon) = \int_{(1-2\epsilon)^{\frac{1}{\beta\delta}}}^{1} (1+2\epsilon)^{\alpha+\beta\gamma-1} \sum_j g_j(1+2\epsilon)d(1+2\epsilon)$$

$$Jg_j(1-2\epsilon) = \int_{(1-2\epsilon)^{\frac{1}{\beta\delta}}}^{1} (1+2\epsilon)^{\alpha+\beta\gamma-1} \sum_j g_j(1+2\epsilon)d(1+2\epsilon)$$

for $\frac{1}{2} < \epsilon < 0$, $\frac{1}{2} < \epsilon < 0$, and

$$J'g_j(1-2\epsilon) = (1-2\epsilon)^{\alpha+\beta\gamma-1} \int_0^{(1-2\epsilon)^{\beta\delta}} \sum_j g_j(1+2\epsilon)d(1+2\epsilon) \quad \text{for } \frac{1}{2} < \epsilon < 0$$

for $g_j \in \mathcal{M}_+(0,1)$ $g_j \in \mathcal{M}_+(0,1)$. Denote by $\|\cdot\|_{X_{J'}(0,1)}$ $\|\cdot\|_{X_{J'}(0,1)}$ the functional given by

$$\|\textstyle\sum_j g_j\|_{X_{J'}(0,1)} = \|J' \textstyle\sum_j g_j^*\|_{X(0,1)}$$

$\|\sum_j g_j\|_{X_{,}'(0,1)} = \|J' \sum_j g_j^*\|_{X(0,1)}$ for $g_j \in \mathcal{M}_+(0,1)$. $g_j \in \mathcal{M}_+(0,1)$.

Then the first two inequalities in (3.14) guarantee that (3.3) holds with

$\omega\left(\frac{1}{2} < \epsilon < 0\right) = (1-2\epsilon)^{\alpha+\beta\gamma-1}$

$\omega\left(\frac{1}{2} < \epsilon < 0\right) = (1-2\epsilon)^{\alpha+\beta\gamma-1}$ and $1-\epsilon = \beta\delta.1-\epsilon = \beta\delta$.

Therefore, by proposition 3.1, the functional $\|\cdot\|_{X_{j}'(0,1)}\|\cdot\|_{X_{j}'(0,1)}$ is a rearrangement-invariant function norm, and

$$\left\|J\sum_j f_j\right\|_{X_{j}'^{\prime}(0,1)} \leq \sum_j \|f_j\|_{X'(0,1)} \qquad (3.16)$$

for $g_j \in \mathcal{M}_+(0,1) g_j \in \mathcal{M}_+(0,1)$. Set $\theta = 1 - \frac{1-(\alpha+\beta\gamma)}{\beta\delta}$ $\theta = 1 - \frac{1-(\alpha+\beta\gamma)}{\beta\delta}$ and $\eta = 1 - \frac{1-\gamma}{\delta}.\eta = 1 - \frac{1-\gamma}{\delta}$.

Assumption (3.14) ensures that $0 \leq \eta \leq \theta < 1.0 \leq \eta \leq \theta < 1$.

Therefore,

$$\begin{aligned}
(1-2\epsilon)^{-\eta} \sup_{1-2\epsilon \leq 1+2\epsilon \leq 1} (1+2\epsilon)^{\eta} g_j^*(1+2\epsilon) \\
= (1-2\epsilon)^{-\eta} \sup_{1-2\epsilon \leq 1+2\epsilon \leq 1} (1+2\epsilon)^{\eta-\theta}(1+2\epsilon)^{\theta} g_j^*(1+2\epsilon) \\
\leq (1-2\epsilon)^{-\theta} \sup_{1-2\epsilon \leq 1+2\epsilon \leq 1} (1+2\epsilon)^{\theta} g_j^*(1+2\epsilon) \qquad (3.17)
\end{aligned}$$

for $\frac{1}{2} < \epsilon < 0$ for $\frac{1}{2} < \epsilon < 0$ and $g_j \in \mathcal{M}_+(0,1) g_j \in \mathcal{M}_+(0,1)$.

We claim that there exists a constant

$C = C(\alpha,\beta,\gamma,\delta) C = C(\alpha,\beta,\gamma,\delta)$ such that

$$\left\|(1-2\epsilon)^{-\theta} \sup_{1-2\epsilon\leq 1+2\epsilon\leq 1} (1+2\epsilon)^{\theta} \sum_j g_j^*(1+2\epsilon)\right\|_{X_{J'}(0,1)}$$

$$\leq C \sum_j \|g_j\|_{X_{J'}(0,1)} \qquad (3.18)$$

for $g_j \in \mathcal{M}_+(0,1) g_j \in \mathcal{M}_+(0,1)$. To verify this claim, fix any such function g, and begin by observing that, by the definition of the norm $\|\cdot\|_{X_{J'}(0,1)}$,$\|\cdot\|_{X_{J'}(0,1)}$,we have that

$$\left\|(1-2\epsilon)^{-\theta} \sup_{1-2\epsilon\leq 1+2\epsilon\leq 1} (1+2\epsilon)^{\theta} \sum_j g_j^*(1+2\epsilon)\right\|_{X_{J'}(0,1)}$$

$$= \Bigg\|(1$$

$$-2\epsilon)^{\alpha+\beta\gamma-1} \int_0^{(1-2\epsilon)^{\beta\delta}} (1+2\epsilon)^{-\theta} \sup_{1+2\epsilon\leq r\leq 1} \left(r^{\theta} \sum_j g_j^*(r)\right) d(1$$

$$+2\epsilon)\Bigg\|_{X(0,1)} \qquad (3.19)$$

Next,

$$\int_0^{(1-2\epsilon)^{\beta\delta}} (1+2\epsilon)^{-\theta} \sup_{1+2\epsilon\le r\le 1} \left(r^{\theta} \sum_j g_j^*(r) \right) d(1+2\epsilon)$$

$$\le \int_0^{(1-2\epsilon)^{\beta\delta}} (1+2\epsilon)^{-\theta} \sup_{1+2\epsilon\le r\le (1-2\epsilon)^{\beta\delta}} \left(r^{\theta} \sum_j g_j^*(r) \right) d(1+2\epsilon)$$

$$+ \sup_{(1-2\epsilon)^{\beta\delta}\le r\le 1} r^{\theta} g_j^*(r) \int_0^{(1-2\epsilon)^{\beta\delta}} (1+2\epsilon)^{-\theta} d$$

$$= \int_0^{(1-2\epsilon)^{\beta\delta}} (1+2\epsilon)^{-\theta} \sup_{1+2\epsilon\le r\le (1-2\epsilon)^{\beta\delta}} \left(r^{\theta} \sum_j g_j^*(r) \right) d(1+2\epsilon)$$

$$+ \frac{1}{1-\theta} (1-2\epsilon)^{\beta\delta(1-\theta)} \sup_{(1-2\epsilon)^{\beta\delta}\le r\le 1} r^{\theta} g_j^*(r)$$

for $\frac{1}{2} < \epsilon < 0. \frac{1}{2} < \epsilon < 0.$ By inequality (3.11),

$$\int_0^{(1-2\epsilon)^{\beta\delta}} (1+2\epsilon)^{-\theta} \sup_{1+2\epsilon\le r\le (1-2\epsilon)^{\beta\delta}} \left(r^{\theta} \sum_j g_j^*(r) \right) d(1+2\epsilon) \le$$

$$C \int_0^{(1-2\epsilon)^{\beta\delta}} \sum_j g_j^*(1+2\epsilon) d(1+2\epsilon)$$

$$\int_0^{(1-2\epsilon)^{\beta\delta}} (1+2\epsilon)^{-\theta} \sup_{1+2\epsilon\le r\le (1-2\epsilon)^{\beta\delta}} \left(r^{\theta} \sum_j g_j^*(r) \right) d(1+2\epsilon) \le$$

$$C \int_0^{(1-2\epsilon)^{\beta\delta}} \sum_j g_j^*(1+2\epsilon) d(1+2\epsilon)$$

for $\frac{1}{2} < \epsilon < 0, \frac{1}{2} < \epsilon < 0,$

for some constant $C = C(\alpha, \beta, \gamma, \delta). C = C(\alpha, \beta, \gamma, \delta)$. Thus, by the definition of θ, θ,

$$\int_0^{(1-2\epsilon)^{\beta\delta}} (1+2\epsilon)^{-\theta} \sup_{1+2\epsilon\leq r\leq 1} (r^{\theta}) d(1+2\epsilon)$$
$$\leq C'\left(\int_0^{(1-2\epsilon)^{\beta\delta}} \sum_j g_j^*(1+2\epsilon)d(1+2\epsilon)\right.$$
$$\left.+(1-2\epsilon)^{1-(\alpha+\beta\gamma)} \sup_{(1-2\epsilon)^{\beta\delta}\leq r\leq 1} r^{\theta} g_j^*(r)\right)$$

for $\frac{1}{2} < \epsilon < 0,\frac{1}{2} < \epsilon < 0$,where $C' = \max\left\{C,\frac{1}{1-\theta}\right\}$. $C' = \max\left\{C,\frac{1}{1-\theta}\right\}$. On making use of this inequality in (3.19), we obtain that

$$\left\|(1-2\epsilon)^{-\theta} \sup_{1-2\epsilon\leq 1+2\epsilon\leq 1} (1+2\epsilon)^{\theta} \sum_j g_j^*(1+2\epsilon)\right\|_{X_{J'}(0,1)}$$
$$\leq C'\left\|(1-2\epsilon)^{\alpha+\beta\gamma-1} \int_0^{(1-2\epsilon)^{\beta\delta}} \sum_j g_j^*(1+2\epsilon)d(1+2\epsilon)\right.$$
$$\left.+\sup_{(1-2\epsilon)^{\beta\delta}\leq r\leq 1} r^{\theta} g_j^*(r)\right\|_{X(0,1)}$$
$$= C'\left\|(1-2\epsilon)^{\theta\beta\delta} \sum_j g_j^{**}\left((1-2\epsilon)^{\beta\delta}\right) + \sup_{(1-2\epsilon)^{\beta\delta}\leq r\leq 1} r^{\theta} g_j^*(r)\right\|_{X(0,1)}$$
$$\leq 2C'\left\|\sup_{1-2\epsilon\leq r\leq 1} r^{\theta\beta\delta} \sum_j g_j^{**}\left(r^{\beta\delta}\right)\right\|_{X(0,1)} . \quad (3.20)$$

By inequality (3.10), with $\theta\theta$ replaced by $\theta\beta\delta,\theta\beta\delta,$ and (2.2), there exists a constant $C = C(\alpha,\beta,\gamma,\delta)C = C(\alpha,\beta,\gamma,\delta)$ such that

$$\int_0^{1-2\epsilon} \sup_{1+2\epsilon\leq\tau\leq 1} \tau^{\theta\beta\delta} \sum_j g_j^{**}\left(\tau^{\beta\delta}\right) d(1+2\epsilon)$$

$$\leq C\int_0^{1-2\epsilon} (1+2\epsilon)^{\theta\beta\delta} \sum_j g_j^{**}\left(s^{\beta\delta}\right) d(1+2\epsilon)$$

$$\leq C\int_0^{1-2\epsilon} \left[(1+2\epsilon)^{\theta\beta\delta} \sum_j g_j^{**}\left((1+2\epsilon)^{\beta\delta}\right)\right]^* (r)dr \quad (3.21)$$

for $\frac{1}{2} < \epsilon < 0.\frac{1}{2} < \epsilon < 0.$ On the other hand, by the monotonicity of the function $\tau \mapsto g_i^{**}(\tau^{\beta\delta})\tau \mapsto g_i^{**}(\tau^{\beta\delta})$ and (2.2) again,

$$(1-2\epsilon) \sup_{1-2\epsilon\leq\tau\leq 1} \tau^{\theta\beta\delta} g_j^{**}\left(\tau^{\beta\delta}\right) = (\theta\beta\delta+1)(1-2\epsilon) \sup_{1-2\epsilon\leq\tau\leq 1} g_j^{**}\left(\tau^{\beta\delta}\right)\frac{1}{\tau}\int_0^{\tau} r^{\theta\beta\delta} dr$$

$$\leq (\theta\beta\delta+1)(1-2\epsilon) \sup_{1-2\epsilon\leq\tau\leq 1} \frac{1}{\tau}\int_0^{\tau} r^{\theta\beta\delta} \sum_j g_j^{**}\left(r^{\beta\delta}\right) dr$$

$$\leq (\theta\beta\delta+1)(1-2\epsilon) \sup_{1-2\epsilon\leq\tau\leq 1} \frac{1}{\tau}\int_0^{\tau} \left[(1+2\epsilon)^{\theta\beta\delta} \sum_j g_j^{**}\left((1+2\epsilon)^{\beta\delta}\right)\right]^* (r)dr$$

$$= (\theta\beta\delta+1)\int_0^{1-2\epsilon} \left[(1+2\epsilon)^{\theta\beta\delta} \sum_j g_j^{**}\left((1+2\epsilon)^{\beta\delta}\right)\right]^* (r)dr \qquad (3.22)$$

for $\frac{1}{2} < \epsilon < 0.\frac{1}{2} < \epsilon < 0.$

Owing to (3.21) and (3.22),

$$\int_0^{1-2\epsilon}\left[\sup_{1+2\epsilon\le\tau\le 1}\tau^{\theta\beta\delta}\sum_j g_j^{**}\left(\tau^{\beta\delta}\right)\right]^*(r)dr=\int_0^{1-2\epsilon}\sup_{1+2\epsilon\le\tau\le 1}\tau^{\theta\beta\delta}\sum_j g_j^{**}\left(\tau^{\beta\delta}\right)d(1+2\epsilon)\le\int_0^{1-2\epsilon}\sup_{1+2\epsilon\le\tau\le 1-2\epsilon}\tau^{\theta\beta\delta}\sum_j g_j^{**}\left(\tau^{\beta\delta}\right)d(1+2\epsilon)+(1-2\epsilon)\sup_{1-2\epsilon\le\tau\le 1}\tau^{\theta\beta\delta}g_j^{**}\left(\tau^{\beta\delta}\right)\le C\int_0^{1-2\epsilon}\left[(1+2\epsilon)^{\theta\beta\delta}\sum_j g_j^{**}\left((1+2\epsilon)^{\beta\delta}\right)\right]^*(r)dr$$

$$\int_0^{1-2\epsilon}\left[\sup_{1+2\epsilon\le\tau\le 1}\tau^{\theta\beta\delta}\sum_j g_j^{**}\left(\tau^{\beta\delta}\right)\right]^*(r)dr=\int_0^{1-2\epsilon}\sup_{1+2\epsilon\le\tau\le 1}\tau^{\theta\beta\delta}\sum_j g_j^{**}\left(\tau^{\beta\delta}\right)d(1+2\epsilon)\le\int_0^{1-2\epsilon}\sup_{1+2\epsilon\le\tau\le 1-2\epsilon}\tau^{\theta\beta\delta}\sum_j g_j^{**}\left(\tau^{\beta\delta}\right)d(1+2\epsilon)+(1-2\epsilon)\sup_{1-2\epsilon\le\tau\le 1}\tau^{\theta\beta\delta}g_j^{**}\left(\tau^{\beta\delta}\right)\le C\int_0^{1-2\epsilon}\left[(1+2\epsilon)^{\theta\beta\delta}\sum_j g_j^{**}\left((1+2\epsilon)^{\beta\delta}\right)\right]^*(r)dr$$

for $\frac{1}{2}<\epsilon<0.\frac{1}{2}<\epsilon<0.$

for some constant $C=C(\alpha,\beta,\gamma,\delta).C=C(\alpha,\beta,\gamma,\delta)$. Hence, by the Hardy-Littlewood-Pólya principle,

$$\left\|\sup_{1-2\epsilon\le 1+2\epsilon\le 1}(1+2\epsilon)^{\theta\beta\delta}\sum_j g_j^{**}\left((1+2\epsilon)^{\beta\delta}\right)\right\|_{X(0,1)}$$

$$\le C\left\|(1-2\epsilon)^{\theta\beta\delta}\sum_j g_j^{**}\left((1-2\epsilon)^{\beta\delta}\right)\right\|_{X(0,1)}=C\left\|\sum_j g_j\right\|_{X_{j}'(0,1)},$$

and (3.18) follows from (3.20).

Now, fix any nonnegative function $g_j\in X_{J'}(0,1).g_j\in X_{J'}(0,1)$. By the definition of associate space, (2.3), and Fubini's theorem, we get that

$$\left\|(1-2\epsilon)^{\alpha-1}\int_0^{(1-2\epsilon)^\beta}\left[\tau^{\gamma-1}\int_0^{\tau^\delta}\sum_j g_j^*(r)dr\right]^*(1+2\epsilon)d(1+2\epsilon)\right\|_{X(0,1)}$$

$$=\sup_{f_j\in\mathcal{M}_+(0,1)\|\sum_j f_j\|_{X'(0,1)}\le 1}\int_0^1\sum_j f_j(1$$

$$-2\epsilon)(1-2\epsilon)^{\alpha-1}\int_0^{(1-2\epsilon)^\beta}\left[\tau^{\gamma-1}\int_0^{\tau^\delta}\sum_j g_j^*(r)dr\right]^*(1+2\epsilon)d(1$$

$$+2\epsilon)d(1-2\epsilon)$$

$$=\sup_{f_j\in\mathcal{M}_+(0,1)\|\sum_j f_j\|_{X'(0,1)}\le 1}\int_0^1\left[\tau^{\gamma-1}\int_0^{\tau^\delta}\sum_j g_j^*(r)dr\right]^*(1$$

$$+2\epsilon)\int_{s^{\frac{1}{\beta}}}^1\sum_j f_j(1-2\epsilon)(1-2\epsilon)^{\alpha-1}d(1-2\epsilon)\,d(1+2\epsilon)\le$$

$$\sup_{f_j\in\mathcal{M}_+(0,1)\|\sum_j f_j\|_{X'(0,1)}\le 1}\int_0^1\left[\tau^{\gamma-1}\int_0^{\tau^\delta}r^{-\eta}\sup_{r\le\rho\le 1}\left(\rho^\eta\sum_j g_j^*(\rho)\right)dr\right]^*(1$$

$$+2\epsilon)\int_{s^{\frac{1}{\beta}}}^1\sum_j f_j(1-2\epsilon)(1-2\epsilon)^{\alpha-1}d(1-2\epsilon)\,ds.\ (3.23)$$

Since $\int_0^{\tau^\delta} r^{-\eta}dr=\frac{\tau^{(1-\eta)\delta}}{1-\eta}=\frac{\tau^{1-\gamma}}{1-\eta}$, $\int_0^{\tau^\delta} r^{-\eta}dr=\frac{\tau^{(1-\eta)\delta}}{1-\eta}=\frac{\tau^{1-\gamma}}{1-\eta}$, the function

$$(0,1)\ni\tau\mapsto\tau^{\gamma-1}\int_0^{\tau^\delta}r^{-\eta}\sup_{r\le\rho\le 1}\left(\rho^\eta\sum_j g_j^*(\rho)\right)dr$$

is non-increasing on $(0,1)(0,1)$, inasmuch as it is a constant multiple of the intergral mean over $(0,\tau^{\delta})(0,\tau^{\delta})$ of the non-increasing function$r\mapsto \sup\limits_{r\le\rho\le 1}\rho^{\eta}g_j^*(\rho)r\mapsto \sup\limits_{r\le\rho\le 1}\rho^{\eta}g_j^*(\rho)$ with respect to measure $r^{-\eta}dr.\,r^{-\eta}dr.$

Consequently, by (3.23)

$$\left\|(1-2\epsilon)^{\alpha-1}\int_0^{(1-2\epsilon)^{\beta}}\left[\tau^{\gamma-1}\int_0^{\tau^{\delta}}\sum_j g_j^*(r)dr\right]^*(1+2\epsilon)d(1+2\epsilon)\right\|_{X(0,1)} \le$$

$$\sup_{f_j\in\mathcal{M}_+(0,1)\|f_j\|_{X'(0,1)}\le 1}\int_0^1(1+2\epsilon)^{\gamma-1}\left(\int_{(1+2\epsilon)^{\frac{1}{\beta}}}^1\sum_j f_j\,(1-2\epsilon)(1-2\epsilon)^{\alpha-1}d(1-2\epsilon)\,d(1+2\epsilon)\right)\int_0^{(1+2\epsilon)^{\delta}}r^{-\eta}\sup_{r\le\rho\le 1}\left(\rho^{\eta}\sum_j g_j^*(\rho)\right)drd(1+2\epsilon)=$$

$$\sup_{f_j\in\mathcal{M}_+(0,1)\|\sum_j f_j\|_{X'(0,1)}\le 1}\int_0^1 r^{-\eta}\sup_{r\le\rho\le 1}\rho^{\eta}\sum_j g_j^*(\rho)\int_{r^{\frac{1}{\delta}}}^1(1+2\epsilon)^{\gamma-1}\int_{(1+2\epsilon)^{\frac{1}{\beta}}}^1\sum_j f_j\,(1-2\epsilon)(1-2\epsilon)^{\alpha-1}d(1-2\epsilon)\,d(1+2\epsilon)\,dr\le$$

$$\sup_{f_j\in\mathcal{M}_+(0,1)\|f_j\|_{X'(0,1)}\le 1}\left\|r^{-\eta}\sup_{r\le\rho\le 1}\left(\rho^{\eta}\sum_j g_j^*(\rho)\right)\right\|_{X_j'(0,1)}.\ (3.24)$$

$$\left\|(1-2\epsilon)^{\alpha-1}\int_0^{(1-2\epsilon)^{\beta}}\left[\tau^{\gamma-1}\int_0^{\tau^{\delta}}\sum_j g_j^*(r)dr\right]^*(1+2\epsilon)d(1+2\epsilon)\right\|_{X(0,1)} \le$$

$$\sup_{f_j\in\mathcal{M}_+(0,1)\|f_j\|_{X'(0,1)}\le 1}\int_0^1(1+2\epsilon)^{\gamma-1}\left(\int_{(1+2\epsilon)^{\frac{1}{\beta}}}^1\sum_j f_j\,(1-2\epsilon)(1-2\epsilon)^{\alpha-1}d(1-2\epsilon)\,d(1+2\epsilon)\right)\int_0^{(1+2\epsilon)^{\delta}}r^{-\eta}\sup_{r\le\rho\le 1}\left(\rho^{\eta}\sum_j g_j^*(\rho)\right)drd(1+2\epsilon)=$$

$$\sup_{f_j\in\mathcal{M}_+(0,1)\|\sum_j f_j\|_{X'(0,1)}\le 1}\int_0^1 r^{-\eta}\sup_{r\le\rho\le 1}\rho^{\eta}\sum_j g_j^*(\rho)\int_{r^{\frac{1}{\delta}}}^1(1+2\epsilon)^{\gamma-1}\int_{(1+2\epsilon)^{\frac{1}{\beta}}}^1\sum_j f_j\,(1-2\epsilon)(1-2\epsilon)^{\alpha-1}d(1-2\epsilon)\,d(1+2\epsilon)\,dr\le$$

$$\sup_{f_j\in\mathcal{M}_+(0,1)\|f_j\|_{X'(0,1)}\le 1}\left\|r^{-\eta}\sup_{r\le\rho\le 1}\left(\rho^{\eta}\sum_j g_j^*(\rho)\right)\right\|_{X_j'(0,1)}.\ (3.24)$$

By (3.17) and (3.18),

$$\left\| r^{-\eta} \sup_{r\leq\rho\leq 1}\left(\rho^{\eta}\sum_j g_j^*(\rho)\right)\right\|_{X_{J'}(0,1)} \leq \left\| r^{-\theta} \sup_{r\leq\rho\leq 1} \rho^{\theta}\sum_j g_j^*(\rho)\right\|_{X_{J'}(0,1)}$$

$$\leq C\sum_j \|g_j\|_{X_{J'}(0,1)}. (3.25)$$

On the other hand, by (3.16),

$$\left\|\int_{r^{\frac{1}{\delta}}}^{1} r^{\gamma-1}\int_{(1+2\epsilon)^{\frac{1}{\beta}}}^{1}\Sigma_j f_j\,(1-2\epsilon)(1-2\epsilon)^{\alpha-1}d(1-2\epsilon)\,dr\right\|_{X'_{J'}(0,1)} =$$

$$\beta\left\|\int_{(1+2\epsilon)^{\frac{1}{\beta\delta}}}^{1}\tau^{\beta\gamma-1}\int_{\tau}^{1}\Sigma_j f_j\,(1-2\epsilon)(1-2\epsilon)^{\alpha-1}d(1-2\epsilon)\,d\tau\right\|_{X'_{J'}(0,1)} =$$

$$\beta\left\|\int_{(1+2\epsilon)^{\frac{1}{\beta\delta}}}^{1}\Sigma_j f_j\,((1-2\epsilon))(1-2\epsilon)^{\alpha-1}\int_{(1+2\epsilon)^{\frac{1}{\beta\delta}}}^{1-2\epsilon}\tau^{\beta\gamma-1}d\tau d(1-2\epsilon)\right\|_{X'_{J'}(0,1)} \leq$$

$$\frac{1}{\gamma}\left\|\int_{(1+2\epsilon)^{\frac{1}{\beta\delta}}}^{1}(1-2\epsilon)^{\alpha+\beta\gamma-1}\Sigma_j f_j\,(1-2\epsilon)d(1-2\epsilon)\right\|_{X'_{J'}(0,1)} = \frac{1}{\gamma}\|J\,\Sigma_j f_j\|_{X'_{J'}(0,1)} \leq$$

$$\frac{1}{\gamma}\Sigma_j\|f_j\|_{X'(0,1)} \qquad (3.26)$$

$$\left\|\int_{r^{\frac{1}{\delta}}}^{1} r^{\gamma-1}\int_{(1+2\epsilon)^{\frac{1}{\beta}}}^{1}\Sigma_j f_j\,(1-2\epsilon)(1-2\epsilon)^{\alpha-1}d(1-2\epsilon)\,dr\right\|_{X'_{J'}(0,1)} =$$

$$\beta\left\|\int_{(1+2\epsilon)^{\frac{1}{\beta\delta}}}^{1}\tau^{\beta\gamma-1}\int_{\tau}^{1}\Sigma_j f_j\,(1-2\epsilon)(1-2\epsilon)^{\alpha-1}d(1-2\epsilon)\,d\tau\right\|_{X'_{J'}(0,1)} =$$

$$\beta\left\|\int_{(1+2\epsilon)^{\frac{1}{\beta\delta}}}^{1}\Sigma_j f_j\,((1-2\epsilon))(1-2\epsilon)^{\alpha-1}\int_{(1+2\epsilon)^{\frac{1}{\beta\delta}}}^{1-2\epsilon}\tau^{\beta\gamma-1}d\tau d(1-2\epsilon)\right\|_{X'_{J'}(0,1)} \leq$$

$$\frac{1}{\gamma}\left\|\int_{(1+2\epsilon)^{\frac{1}{\beta\delta}}}^{1}(1-2\epsilon)^{\alpha+\beta\gamma-1}\Sigma_j f_j\,(1-2\epsilon)d(1-2\epsilon)\right\|_{X'_{J'}(0,1)} = \frac{1}{\gamma}\|J\,\Sigma_j f_j\|_{X'_{J'}(0,1)} \leq$$

$$\frac{1}{\gamma}\Sigma_j\|f_j\|_{X'(0,1)} \qquad (3.26)$$

for every nonnegative function $f_j \in X'(0,1) f_j \in X'(0,1)$. It follows from (3.24) ,(3.25) and (3.26) that

$$\sup_{g_j \geq \|\sum_j g_j\|_{X_j'(0,1)} \leq 1} \left\|(1-2\epsilon)^{\alpha-1}\int_0^{(1-2\epsilon)^{\beta}}\left[\tau^{\gamma-1}\int_0^{\tau^{\delta}}\sum_j g_j^*(r)dr\right]^*(1+2\epsilon)d(1+2\epsilon)\right\|_{X(0,1)} \leq \frac{C}{\gamma}$$

$$\sup_{g_j \geq \|\sum_j g_j\|_{X_j'(0,1)} \leq 1} \left\|(1-2\epsilon)^{\alpha-1}\int_0^{(1-2\epsilon)^{\beta}}\left[\tau^{\gamma-1}\int_0^{\tau^{\delta}}\sum_j g_j^*(r)dr\right]^*(1+2\epsilon)d(1+2\epsilon)\right\|_{X(0,1)} \leq \frac{C}{\gamma}$$

.

Hence, inequality (3.15) follows from the definition of the norm $\|\cdot\|_{X'(0,1)}.\|\cdot\|_{X'(0,1)}$.

Let $n \in \mathbb{N} n \in \mathbb{N}$ and let $\|\cdot\|_{Z(0,1)}\|\cdot\|_{Z(0,1)}$ be a rearrangement-invariant function norm. For $j = 0, \dots, n, j = 0, \dots, n$, defined the functional $\|\cdot\|_{Z_j'(0,1)}\|\cdot\|_{Z_j'(0,1)}$ inductively as $\|\cdot\|_{Z_0'(0,1)} = \|\cdot\|_{Z'(0,1)}$, $\|\cdot\|_{Z_0'(0,1)} = \|\cdot\|_{Z'(0,1)}$, and

$$\left\|\sum_j g_j\right\|_{Z_j'(0,1)} = \left\|(1-2\epsilon)^{-\frac{n-j}{n-j+1}}\int_0^{(1-2\epsilon)^{\frac{n-j}{n-j+1}}}\sum_j g_j^*(1+2\epsilon)d(1+2\epsilon)\right\|_{Z_{j-1}'(0,1)} \quad j = 1, \dots, n \quad (3.27)$$

for $g_j \in \mathcal{M}_+(0,1) g_j \in \mathcal{M}_+(0,1)$. Proposition 3.1, applied with $1-\epsilon = \frac{n-j}{n-j+1} 1-\epsilon = \frac{n-j}{n-j+1}$ and $\omega(1-2\epsilon) = (1-2\epsilon)^{-\left(\frac{n-j}{n-j+1}\right)}$, $\omega(1-2\epsilon) = (1-2\epsilon)^{-\left(\frac{n-j}{n-j+1}\right)}$, guarantees that, for each

$j = 0, \dots, n, j = 0, \dots, n,$the functional $\|\cdot\|_{Z_j'(0,1)}\|\cdot\|_{Z_j'(0,1)}$ is a rearrangement-invariant function norm.

Theorem 3.5. Let $n \in \mathbb{N} n \in \mathbb{N}$ and let $\|\cdot\|_{Z(0,1)}\|\cdot\|_{Z(0,1)}$ be a rearrangement-invariant function norm.Then for every$j = 0, \dots, n,$ $j = 0, \dots, n,$ there exists a positiveconstant $C_j,C_j,$ depending only on j j and $n n$, such that

$$\left\|\sum_j g_j\right\|_{Z_j'(0,1)}$$

$$\leq C_j \left\|(1-2\epsilon)^{-1+\frac{j}{n}} \int_0^{(1-2\epsilon)^{1-\frac{j}{n}}} \sum_j g_j^*(1+2\epsilon)d(1+2\epsilon)\right\|_{Z'(0,1)} \tag{3.28}$$

for $g_j \in \mathcal{M}_+(0,1) g_j \in \mathcal{M}_+(0,1),$ where $\|\cdot\|_{Z_j'(0,1)}\|\cdot\|_{Z_j'(0,1)}$ is a rearrangement-invariant function norm defined by (3.27).

Proof. We argue by finite induction. The assertion for $j = 0 j = 0$ holds with $C_1 = 1,\ C_1 = 1,$ thanks to the fact that $Z_0(0,1) = Z(0,1)$ $Z_0(0,1) = Z(0,1)$ and that $g_j^* \leq g_j^{**}.g_j^* \leq g_j^{**}$. Assume next that the claim is true for some $j = 0, 1, \dots, n-2. j = 0, 1, \dots, n-2.$ Then, by Theorem 3.4 applied with

$\alpha = \frac{j}{n}, \beta = 1 - \frac{j}{n}, \gamma = \frac{1}{n-j} \alpha = \frac{j}{n}, \beta = 1 - \frac{j}{n}, \gamma = \frac{1}{n-j}$ and $= 1 - \frac{1}{n-j} = 1 - \frac{1}{n-j}$, we get that

$$\left\|\sum_j g_j\right\|_{Z'_{j+1}(0,1)} = \left\|(1-2\epsilon)^{\frac{n-j-1}{n-j}} \int_0^{(1-2\epsilon)^{\frac{n-j-1}{n-j}}} \sum_j g_j^*(1+2\epsilon)d(1+2\epsilon)\right\|_{Z'_j(0,1)}$$

$$\leq C_j \left\|(1-2\epsilon)^{\frac{j}{n}-1} \int_0^{(1-2\epsilon)^{1-\frac{j}{n}}} \left[\tau^{-\frac{n-j-1}{n-j}} \int_0^{(1-2\epsilon)^{\frac{n-j-1}{n-j}}} \sum_j g_j^*(r)dr\right]^* (1 + 2\epsilon)d(1+2\epsilon)\right\|_{Z'(0,1)}$$

$$\leq C_j C \left\|(1-2\epsilon)^{\frac{j+1}{n}-1} \int_0^{(1-2\epsilon)^{1-\frac{j+1}{n}}} \sum_j g_j^*(1+2\epsilon)d(1+2\epsilon)\right\|_{Z'(0,1)}$$

for every $g_j \in \mathcal{M}_+(0,1) g_j \in \mathcal{M}_+(0,1)$, where CC is the constant appearing in (3.15). This establishes (3.28) for $j = 1, \dots, n-1. j = 1, \dots, n-1$. It remains to consider the case when $j = n. j = n$. In this case, equation (3.27) yields

$$\|\Sigma_j g_j\|_{Z'_n(0,1)} = \|1\|_{Z'_{n-1}(0,1)} \int_0^1 \Sigma_j g_j^*(1+2\epsilon)d(1+2\epsilon)$$

$$\|\Sigma_j g_j\|_{Z'_n(0,1)} = \|1\|_{Z'_{n-1}(0,1)} \int_0^1 \Sigma_j g_j^*(1+2\epsilon)d(1+2\epsilon)$$

for $g_j \in \mathcal{M}_+(0,1) g_j \in \mathcal{M}_+(0,1)$. On the other hand, for $j = n$, $j = n$, we have

$$\left\| (1-2\epsilon)^{\frac{j}{n}-1} \int_0^{1-\frac{j}{n}} \sum_j g_j^*(1+2\epsilon)d(1+2\epsilon) \right\|_{Z'(0,1)}$$
$$= \|1\|_{Z'(0,1)} \int_0^1 \sum_j g_j^*(1+2\epsilon)d(1+2\epsilon)$$

for $g_j \in \mathcal{M}_+(0,1) g_j \in \mathcal{M}_+(0,1)$, whence (3.28) follows.The proof is complete.

4 . Proofs of the main results

Our approach makes use of reduction principles for Sobolev embeddings on the whole of $\lambda\lambda$, and for trace embeddings on $\lambda_{n-1}\lambda_{n-1}$. They are stated in Theorem 4.1and in Theorem 4.2, respectively, below.

Theorem 4.1. Let $\lambda\lambda$ be a bounded open set with the cone property in $\mathbb{R}^n, n \geq 2, \mathbb{R}^n, n \geq 2$, and let $m \in \mathbb{N}. m \in \mathbb{N}$.Assume that $\|\cdot\|_{X(0,1)}\|\cdot\|_{X(0,1)}$and $\|\cdot\|_{Y(0,1)}\|\cdot\|_{Y(0,1)}$ are rearrangement-invariant function norms. Then the Sobolev embedding

$$W^m X(\lambda) \to Y(\lambda) \qquad (4.1)$$

holds if and only if the Hardy type inequality

$$\left\| \int_{1-2\epsilon}^1 \sum_j f_j(1+2\epsilon)(1+2\epsilon)^{-1+\frac{m}{n}} d(1+2\epsilon) \right\|_{Y(0,1)} \leq C \sum_j \|f_j\|_{X(0,1)} \qquad (4.2)$$

holds for some constant CC and for every nonnegative function $f_j \in X(0,1) f_j \in X(0,1)$.

Theorem 4.1 follows from a special case of ([17] Theorem 6.1, via ([26] Corollary 5.2.1/3). In the case when $\lambda\lambda$ is a Lipschitz domain and $m \leq n-1 m \leq n-1$ it was proved in [25].

Theorem 4.2. Let $\lambda\lambda$ be a bounded open set with the cone property in $\mathbb{R}^n, n \geq 2, \mathbb{R}^n, n \geq 2$, and let $m \in \mathbb{N}. m \in \mathbb{N}$. Assume that $\|\cdot\|_{X(0,1)} \|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)} \|\cdot\|_{Y(0,1)}$ are rearrangement-invariant function norms.Then the Sobolev trace embedding

$$Tr: W^1 X(\lambda) \to Y(\lambda_{n-1}) \qquad (4.3)$$

holds if and only if the Hardy type inequality

$$\left\| \int_{(1-2\epsilon)^{\frac{n}{n-1}}}^{1} \sum_j f_j (1+2\epsilon)(1+2\epsilon)^{-1+\frac{1}{n}} d(1+2\epsilon) \right\|_{Y(0,1)} \leq C \sum_j \|f_j\|_{X(0,1)} \qquad (4.4)$$

holds for some constant CC and for every nonnegative function $f_j \in X(0,1). f_j \in X(0,1)$.

A version of Theorem 4.2, where $\lambda_{n-1}\lambda_{n-1}$ is replaced with $\partial\lambda, \partial\lambda$, can be found ([14] Theorem 3.1), and is based on an interpolation argument which makes use of Peetre's K-functional. The proof of the fact that (4.4) implies (4.3) is completely analogous. One has just to replace the endpoint inequalities exploited in the interpolation argument of [14] which the following inequalities for the trace operator $TrTr$ on $\lambda_{n-1}: \|Tr\, u\|_{L^1(\lambda_{n-1})} \leq C\|\, u\|_{W^{1,1}(\lambda)}$ $\lambda_{n-1}: \|Tr\, u\|_{L^1(\lambda_{n-1})} \leq C\|\, u\|_{W^{1,1}(\lambda)}$ for every $u \in W^{1,1}(\lambda)$, $u \in W^{1,1}(\lambda)$, and $\|Tr\, u\|_{L^\infty(\lambda_{n-1})} \leq C\|\, u\|_{W^1 L^{n,1}(\lambda)}$ $\|Tr\, u\|_{L^\infty(\lambda_{n-1})} \leq C\|\, u\|_{W^1 L^{n,1}(\lambda)}$ for every $u \in W^1 L^{n,1}(\lambda)$, $u \in W^1 L^{n,1}(\lambda)$, where $C = C(\lambda, \lambda_{n-1}). C = C(\lambda, \lambda_{n-1})$. Note that the former inequality is just a special case of (1.2), whereas the latter holds owing to the embedding $W^1 L^{n,1}(\lambda) \to C(\lambda)$ $W^1 L^{n,1}(\lambda) \to C(\lambda)$, the space of continuous bounded functions in $\lambda\lambda$ (see for instance ([15] Remark 3.10).The fact that (4.3) implies (4.4) will not be used in the proof of Theorem 1.3. In fact, it follows on specializing a more general argument in the proof of the theorem.

Proof Theorem1.3. The equivalence of (i) and (ii) is the content of Proposition 2.2. Thus, only The equivalence of (i) and (ii) has to be established. Let us first show that (iii) implies (i). We preliminarily observe that it suffices to prove such implication in the case when $\lambda\lambda$ is a ball $\mathcal{B}\,\mathcal{B}$. Indeed, any bounded open set with the cone property can be decomposed into the finite union of Lipschitz domains ([2] Theorem 4.8). It is then easily seen that we may assume that $\lambda\lambda$ is a Lipschitz domain .Next, any Lipschitz domain is an extension domain, both for $W^{m,1}(\lambda)W^{m,1}(\lambda)$ and for$W^{m,\infty}(\lambda)W^{m,\infty}(\lambda)$ ([35] Theorem 5). By ([18] Theorem 4.1), given any ball $\mathcal{B}\supset\bar{\lambda}$ $\mathcal{B}\supset\bar{\lambda}$, there exists a linear bounded extension operator $\epsilon\epsilon$ such that $\epsilon\colon W^mX(\lambda)\to W^mX(\mathcal{B})\epsilon\colon W^mX(\lambda)\to W^mX(\mathcal{B})$for any rearrangement-invariant function norm$\|\cdot\|_{X(0,1)},\|\cdot\|_{X(0,1)}$, with norm independent of $\|\cdot\|_{X(0,1)}.\|\cdot\|_{X(0,1)}$. Namely, $\epsilon u = u\ \epsilon u = u$ in λ,λ,

and there exists a constant $C = C(\lambda,\mathcal{B},m)C = C(\lambda,\mathcal{B},m)$ such that

$$\|\,\epsilon u\|_{W^mX(\mathcal{B})} \leq C\|u\|_{W^mX(\lambda)} \qquad (4.5)$$

for every $u\in W^mX(\lambda).\ u\in W^mX(\lambda)$.Given any dd-dimensional hyperplane such that $\lambda_d\neq\emptyset,\lambda_d\neq\emptyset$, let us denote by $Tr_{\mathcal{B}}Tr_{\mathcal{B}}$ the trace operator on $\mathcal{B}_d\mathcal{B}_d$ acting on functions in $W^{m,1}(\mathcal{B}),W^{m,1}(\mathcal{B})$, and by $Tr_\lambda Tr_\lambda$ the trace operator on $\lambda_d\lambda_d$ acting on functions in $W^{m,1}(\lambda).\ W^{m,1}(\lambda)$. One has that $\lambda_d\subset\mathcal{B}_d,\ \lambda_d\subset\mathcal{B}_d$, and

$Tr_{\mathcal{B}}\epsilon u = Tr_{\lambda} u \quad \mathcal{H}^d - \text{a.e.} \,(4.6)$

in $\lambda_d.\lambda_d$.Now, assume that $\|\cdot\|_{X(0,1)}\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}\|\cdot\|_{Y(0,1)}$ are rearrangement-invariant function norm such that (i) holds in $\mathcal{B}$ $\mathcal{B}$, namely

$$\|Tr_{\mathcal{B}} v\|_{Y(\mathcal{B}_d)} \leq C\|v\|_{W^m X(\mathcal{B})} \qquad (4.7)$$

for some constant CC and for every $v \in W^m X(\mathcal{B}).v \in W^m X(\mathcal{B})$. Thus, by (4.6) (coupled with the boundedness of the dilation operator on rearrangement-invariant spaces), (4.7) and (4.5), there exist constants $C, C', C''C, C', C''$ such that

$$\|Tr_{\lambda} u\|_{Y(\lambda_d)} \leq C\|Tr_{\mathcal{B}}\epsilon u\|_{Y(\mathcal{B}_d)} \leq C'\|\epsilon u\|_{W^m X(\mathcal{B})} \leq C''\|u\|_{W^m X(\lambda)}$$

for every $u \in W^m X(\lambda) u \in W^m X(\lambda)$.This proves that (i) also holds in $\lambda\lambda$.

We may thus assume,without loss of generality, that $\lambda\lambda$ is a ball, and that $0 \in \lambda_d.0 \in \lambda_d$. Suppose, for the time being, that $m < n.m < n$. We set $k = n - dk = n - d$ and assume first that $k < m.k < m$.There exists a finite sequence of affine $(n-j)(n-j)$-dimensional subspaces of $\mathbb{R}^n, j = 1, \ldots, k$, $\mathbb{R}^n, j = 1, \ldots, k$, such that $0 \in \lambda_d = \lambda_{n-k} \subset \lambda_{n-k+1} \subset \cdots \subset \lambda_{n-1}$. $0 \in \lambda_d = \lambda_{n-k} \subset \lambda_{n-k+1} \subset \cdots \subset \lambda_{n-1}$.

Observe that, since $\lambda\lambda$ is a ball, then $\lambda_{n-j}\lambda_{n-j}$ is an $(n-j)(n-j)$ -dimensional ball for $j = 1, \ldots, k, j = 1, \ldots, k$, and hence an $(n-j)$ $(n-j)$-dimensional Lipschitz domain. Define the functional $\|\cdot\|_{Z'(0,1)}\|\cdot\|_{Z'(0,1)}$ by

$$\left\|\sum_j g_j\right\|_{Z'(0,1)} = \left\|(1-2\epsilon)^{-1+\frac{m-k}{n}} \int_0^{1-2\epsilon} \sum_j g_j^*(1+2\epsilon)\, d(1+2\epsilon)\right\|_{X'(0,1)} \qquad (4.8)$$

for every $g_j \in \mathcal{M}_+(0,1) g_j \in \mathcal{M}_+(0,1)$. By Proposition 3.1, applied with $\epsilon = 0\, \epsilon = 0$ and

$$\omega(1-2\epsilon) = (1-2\epsilon)^{-1+\frac{m-k}{n}}, \omega(1-2\epsilon) = (1-2\epsilon)^{-1+\frac{m-k}{n}},$$

the functional $\|\cdot\|_{Z'(0,1)} \|\cdot\|_{Z'(0,1)}$ is a rearrangement-invariant function norm, and

$$\left\| \int_{1-2\epsilon}^{1} \sum_j f_j\,(1+2\epsilon)(1+2\epsilon)^{-1+\frac{m-k}{n}} d(1+2\epsilon) \right\|_{Z(0,1)} \leq \sum_j \|f_j\|_{X(0,1)}$$

$$\left\| \int_{1-2\epsilon}^{1} \sum_j f_j\,(1+2\epsilon)(1+2\epsilon)^{-1+\frac{m-k}{n}} d(1+2\epsilon) \right\|_{Z(0,1)} \leq \sum_j \|f_j\|_{X(0,1)}$$

for every $f_j \in \mathcal{M}_+(0,1) f_j \in \mathcal{M}_+(0,1)$.

Note that assumption (3.3) is fulfilled, since $< m < m$. By Theorem 4.1, this implies that $W^{m-k}X(\lambda) \to Z(\lambda) W^{m-k}X(\lambda) \to Z(\lambda)$, and hence

$$W^m X(\lambda) \to W^k Z(\lambda). \qquad (4.9)$$

For each $j = 0,1,\dots,k\, j = 0,1,\dots,k$, let $\|\cdot\|_{Z_j'(0,1)} \|\cdot\|_{Z_j'(0,1)}$ be the rearrangement-invariant function norm defined by (3.27). By Proposition 3.1,

applied with $q = \frac{n-j-1}{n-i}\, q = \frac{n-j-1}{n-j}$ and $\omega(1-2\epsilon) = (1-2\epsilon)^{-\frac{n-j-1}{n-j}}$ $\omega(1-2\epsilon) = (1-2\epsilon)^{-\frac{n-j-1}{n-j}}$ for each $j = 0,1,\dots,k-1$, $j = 0,1,\dots,k-1$, we have

$$\left\| \int_{\frac{n-j}{n-j-1}}^{1} \sum_j f_j\,(1+2\epsilon)(1+2\epsilon)^{\frac{n-j-1}{n-j}} d(1+2\epsilon) \right\|_{Z_{j+1}(0,1)} \leq \sum_j \|f_j\|_{Z_j(0,1)} \qquad (4.10)$$

for every $f_j \in \mathcal{M}_+(0,1) f_j \in \mathcal{M}_+(0,1)$. By Theorem 4.2, applied with nn replaced with $n-jn-j$ and $\lambda\lambda$ replaced with λ_{n-j}

λ_{n-j} inequalities (4.10) ensure that, for each $j = 0,1,\dots,k-1$, $j = 0,1,\dots,k-1$,

$$Tr\colon W^1 Z_j(\lambda_{n-j}) \to Z_{j+1}(\lambda_{n-j-1}). \qquad (4.11)$$

Also, one classically has that $Tr\colon W^{m,1}(\lambda) \to W^{m-h,1}(\lambda_\ell)$ $Tr\colon W^{m,1}(\lambda) \to W^{m-h,1}(\lambda_\ell)$ provided that $\ell \geq n-h > 0$ ($\ell \geq n-h > 0$ ([2] Theorem 5.4), and hence, in particular, the trace on $\lambda_d\lambda_d$ of an mm-times weakly differentiable function in $\lambda\lambda$ is an $(m-h)(m-h)$-times weakly differentiable function if $n,h,\ell n,h,\ell$ fufil the above inequality. Thus, iterating embedding (4.11) kk times yields

$$W^k Z(\lambda) \overset{Tr}{\to} W^{k-1} Z_1(\lambda_{n-1}) \overset{Tr}{\to} \dots \overset{Tr}{\to} W^1 Z_{k-1}(\lambda_{d+1}) \overset{Tr}{\to} Z_k(\lambda_d) \qquad (4.12)$$

Coupling (4.9) with (4.12) gives

$$Tr\colon W^m X(\lambda) \to Z_k(\lambda_d). \qquad (4.13)$$

We now defined the functional $\|\cdot\|_{Y_k'(0,1)}\|\cdot\|_{Y_k'(0,1)}$ on $\mathcal{M}_+(0,1)$ $\mathcal{M}_+(0,1)$.

$$\left\|\sum_j g_j\right\|_{Y_k'(0,1)} = \left\|(1-2\epsilon)^{-1+\frac{m}{n}} \int_0^{(1-2\epsilon)^{1-\frac{k}{n}}} \sum_j g_j^*(1+2\epsilon)\,d(1+2\epsilon)\right\|_{X'(0,1)} \qquad (4.14)$$

for $g_j \in \mathcal{M}_+(0,1) g_j \in \mathcal{M}_+(0,1)$. Since, by our assumptions, $\frac{m}{n} - 1 + 1 - \frac{k}{n} = \frac{m+d-n}{n} \geq 0, \frac{m}{n} - 1 + 1 - \frac{k}{n} = \frac{m+d-n}{n} \geq 0,$ condition (3.3) is satisfied for $1-\epsilon = \frac{d}{n} 1-\epsilon = \frac{d}{n}$ and $\omega(1-2\epsilon) = (1-2\epsilon)^{-1+\frac{m}{n}}.\omega(1-2\epsilon) = (1-2\epsilon)^{-1+\frac{m}{n}}.$

Therefore, Proposition 3.1 implies that $\|\cdot\|_{Y_k'(0,1)}\|\cdot\|_{Y_k'(0,1)}$ is a rearrangement-invariant function norm and

$$\left\| \int_{(1-2\epsilon)^{\frac{n}{n-k}}}^{1} \sum_j f_j(1+2\epsilon)(1+2\epsilon)^{\frac{m}{n}-1} d(1+2\epsilon) \right\|_{Y_k(0,1)} \leq \sum_j \|f_j\|_{X(0,1)} \quad (4.15)$$

for every $f_j \in \mathcal{M}_+(0,1) f_j \in \mathcal{M}_+(0,1)$. Moreover, $\|\cdot\|_{Y_k(0,1)}$ $\|\cdot\|_{Y_k(0,1)}$ is the optimal rearrangement-invariant function norm for which (4.15) holds. Since, by (1.12), $\|\cdot\|_{Y(0,1)} \|\cdot\|_{Y(0,1)}$ is also a rearrangement-invariant function norm for which (4.15) holds, the optimality of $\|\cdot\|_{Y_k(0,1)} \|\cdot\|_{Y_k(0,1)}$ implies

$$Y_k(0,1) \to Y(0,1). \quad (4.16)$$

We now claim that

$$Z_k(0,1) \to Y_k(0,1). \quad (4.17)$$

By Theorem 3.5, applied to $j = kj = k$ (observe that $k \leq nk \leq n$),

$$\left\| \sum_j g_j \right\|_{Z_k'(0,1)} = C_k \left\| (1-2\epsilon)^{-1+\frac{k}{n}} \int_0^{(1-2\epsilon)^{1-\frac{k}{n}}} \sum_j g_j^*(1+2\epsilon) d(1+2\epsilon) \right\|_{Z'(0,1)}$$

for every $g_j \in \mathcal{M}_+(0,1) g_j \in \mathcal{M}_+(0,1)$. Therefore, by property (3.2) applied to the identity operator, in order to established (4.17), we just need to show that there exists a positive constant $C' C'$ such that

$$\left\| (1-2\epsilon)^{-1+\frac{k}{n}} \int_0^{(1-2\epsilon)^{1-\frac{k}{n}}} \sum_j g_j^*(1+2\epsilon) d(1+2\epsilon) \right\|_{Z'(0,1)}$$

$$\leq C' \sum_j \|g_j\|_{Y_k'(0,1)} \quad (4.18)$$

for every $g_j \in \mathcal{M}_+(0,1) g_j \in \mathcal{M}_+(0,1)$. By the definition of $\|\cdot\|_{Z'(0,1)}\|\cdot\|_{Z'(0,1)}$ and $\|\cdot\|_{Y_k'(0,1)}\|\cdot\|_{Y_k'(0,1)}$(see (4.8) and (4.14)), inequality (4.18) will in turn follow if we prove that there exists a positive constant $C''C''$ such that

$$\left\| (1-2\epsilon)^{\frac{m-k}{n}-1}\int_0^{1-2\epsilon}\left[\tau^{\frac{k}{n}-1}\int_0^{\tau^{1-\frac{k}{n}}}\sum_j g_j^*(r)dr\right]^*(1+2\epsilon)d(1+2\epsilon)\right\|_{X'(0,1)}$$

$$\le C''\left\|(1-2\epsilon)^{\frac{m}{n}-1}\int_0^{(1-2\epsilon)^{1-\frac{k}{n}}}\sum_j g_j^*(1+2\epsilon)d(1+2\epsilon)\right\|_{X'(0,1)}$$

for every $g_j \in \mathcal{M}_+(0,1) g_j \in \mathcal{M}_+(0,1)$. This is a consequence of Theorem 3.4, applied with

$\alpha = \frac{m-k}{n}, \beta = 1, \gamma = \frac{k}{n}\alpha = \frac{m-k}{n}, \beta = 1, \gamma = \frac{k}{n}$ and $\delta = 1 - \frac{k}{n}$. $\delta = 1 - \frac{k}{n}$. This establishes (4.18), and hence also (4.17). Combining (4.13),(4.17) and (4.16) yields

$W^m X(\lambda) \overset{Tr}{\rightarrow} Z_k(\lambda_d) \rightarrow Y_k(\lambda_d) \rightarrow Y(\lambda_d)$,

$W^m X(\lambda) \overset{Tr}{\rightarrow} Z_k(\lambda_d) \rightarrow Y_k(\lambda_d) \rightarrow Y(\lambda_d)$, and (1.10) follows. Consider next the case when $k = m < nk = m < n$. We define the functionals $\|\cdot\|_{Z_k'(0,1)}\|\cdot\|_{Z_k'(0,1)}$ for $j = 0,1,\dots,kj = 0,1,\dots,k$ as above, save that now we set $\|\cdot\|_{Z_0(0,1)} = \|\cdot\|_{X(0,1)}.\|\cdot\|_{Z_0(0,1)} = \|\cdot\|_{X(0,1)}$. The proof of embeddings (4.13) and (4.16) is the same, and even simpler, since (4.9) holds as an equality. It only remains to prove (4.17). By (3.2), it suffices to verify that there exists a positive constant CC such

$$\left\|\sum_j g_j\right\|_{Z_k'(0,1)} \leq C \left\|(1-2\epsilon)^{-1+\frac{m}{n}} \int_0^{(1-2\epsilon)^{1-\frac{m}{n}}} \sum_j g_j^*(1+2\epsilon) d(1+2\epsilon)\right\|_{X'(0,1)}$$

$$\left\|\sum_j g_j\right\|_{Z_k'(0,1)} \leq C \left\|(1-2\epsilon)^{-1+\frac{m}{n}} \int_0^{(1-2\epsilon)^{1-\frac{m}{n}}} \sum_j g_j^*(1+2\epsilon) d(1+2\epsilon)\right\|_{X'(0,1)}$$

for every $g_j \in \mathcal{M}_+(0,1) g_j \in \mathcal{M}_+(0,1)$. This follows from Theorem 3.5, applied with $j = mj = m$ and $Z(0,1) = X(0,1)$. $Z(0,1) = X(0,1)$. Finally, suppose that $m \geq nm \geq n$. Then $W^m L^1(\lambda) \to C(\lambda) W^m L^1(\lambda) \to C(\lambda)$ (see e.g.[26] Theorem 1.4.5]).Therefore, by (2.6)

$$W^m X(\lambda) \to W^m L^1(\lambda) \overset{Tr}{\to} L^\infty(\lambda_d) \to Y(\lambda_d),$$

$W^m X(\lambda) \to W^m L^1(\lambda) \overset{Tr}{\to} L^\infty(\lambda_d) \to Y(\lambda_d),$ and (i) follows. Let us now show that (i) implies (iii). Suppose that (1.10) holds. Assume first that $m < nm < n$.Without loss of generality, we may assume that $0 \in \lambda_d, 0 \in \lambda_d$, and that $\lambda_d = \lambda \cap \{x = (x',0): x' \in \mathbb{R}^d, 0 \in \mathbb{R}^{n-d}\}$. $\lambda_d = \lambda \cap \{x = (x',0): x' \in \mathbb{R}^d, 0 \in \mathbb{R}^{n-d}\}$.

Let RR be a positive number such that $\overline{B_R(0)} \subset \lambda, \overline{B_R(0)} \subset \lambda$, and let $f: [0,\infty) \to [0,\infty) f: [0,\infty) \to [0,\infty)$ be any locally integrable function. Define the mm-times weakly differentiable function $u: \lambda \to [0,\infty) u: \lambda \to [0,\infty)$ as

$$u(x) = \begin{cases} \int_{\omega_n |x|^n}^{\omega_n R^n} \int_{r_1}^{\omega_n R^n} \cdots \int_{r_{m-1}}^{\omega_n R^n} f(r_m) r_m^{-m+\frac{m}{n}} \, dr_m \ldots dr_1 & \text{in } x \in B_R(0) \\ 0 & \text{otherwise.} \end{cases}$$

We now estimate the quantity $\sum_{k=0}^m |\nabla^k u(x)| \sum_{k=0}^m |\nabla^k u(x)|$, along the lines of ([25] Proof of Theorem A]). Let $g_j\colon[0,\infty)\to[0,\infty)$ $g_j\colon[0,\infty)\to[0,\infty)$ be such that $u(x)=g_j(|x|)u(x)=g_j(|x|)$. An induction argument shows that, for every $\ell\in\mathbb{N}\cup\{0\}$, $\ell\in\mathbb{N}\cup\{0\}$, any $\ell^{th}\ell^{th}$order derivative of uu is a linear combination of terms of the form $\frac{x^\alpha g_j{}^{(j)}(|x|)x^\alpha g_j{}^{(j)}(|x|)}{|x|^k \qquad |x|^k}$, where $|\alpha|=i,k+j-i=\ell|\alpha|=i,k+j-i=\ell$ and $0\le i,j\le\ell$. $0\le i,j\le\ell$.

Here we denote $x^\alpha=x_1^{\alpha_1}\dots x_n^{\alpha_n}x^\alpha=x_1^{\alpha_1}\dots x_n^{\alpha_n}$ for $x\in\mathbb{R}^n$ $x\in\mathbb{R}^n$ and $\alpha\alpha$ being a multi-index. Hence, the absolute value of any $\ell^{th}\ell^{th}$order derivative of uu is domained by a constant multiple of $\sum_{j=0}^{\ell}\left|\frac{g_j{}^{(j)}(|x|)}{|x|^{\ell-j}}\right|\sum_{j=0}^{\ell}\left|\frac{g_j{}^{(j)}(|x|)}{|x|^{\ell-j}}\right|$. Thus, there is a positive constant CC such that

$$\sum_{k=0}^{m}|\nabla^k u(x)|\le C\sum_{\ell=0}^{m}\frac{|g_j{}^{(\ell)}(|x|)|}{|x|^{m-\ell}}$$

$\sum_{k=0}^{m}|\nabla^k u(x)|\le C\sum_{\ell=0}^{m}\frac{|g_j{}^{(\ell)}(|x|)|}{|x|^{m-\ell}}$ for $x\in B_R(0)x\in B_R(0)$.

The function $g_j{}^{(\ell)}(|x|)g_j{}^{(\ell)}(|x|)$ is a linear combination of the expressions

$$|x|^{jn-\ell}\int_{\omega_n|x|^n}^{\omega_nR^n}\int_{(1-2\epsilon)_{j+1}}^{\omega_nR^n}\dots\int_{(1-2\epsilon)_{m-1}}^{\omega_nR^n}\sum_j f_j\left((1-2\epsilon)_m\right)(1-2\epsilon)_m{}^{-m+\frac{m}{n}}d(1-2\epsilon)_m\dots d(1-2\epsilon)_{j+1}$$

for $1\le\ell\le m-11\le\ell\le m-1$ and of the expressions

$$|x|^{jn-m}\int_{\omega_n|x|^n}^{\omega_nR^n}\int_{(1-2\epsilon)_{j+1}}^{\omega_nR^n}\dots\int_{(1-2\epsilon)_{m-1}}^{\omega_nR^n}f_j\left((1-2\epsilon)_m\right)(1-2\epsilon)_m{}^{-m+\frac{m}{n}}d(1-2\epsilon)_m\dots d(1-2\epsilon)_{j+1}$$

$$|x|^{jn-m}\int_{\omega_n|x|^n}^{\omega_nR^n}\int_{(1-2\epsilon)_{j+1}}^{\omega_nR^n}\cdots\int_{(1-2\epsilon)_{m-1}}^{\omega_nR^n} f_j((1-2\epsilon)_m)(1-2\epsilon)_m^{\ -m+\frac{m}{n}}\, d(1-2\epsilon)_m \ldots d(1-2\epsilon)_{j+1}$$

and $f_j(\omega_n|x|^n)\, f_j(\omega_n|x|^n)$ when $\ell = m,\ell = m,$ where $j = 1,2,\ldots,m-1.j = 1,2,\ldots,m-1.$

As a consequence, subsequenct applications of Fubini's theorem and obvious estimates tell us that, $1 \leq \ell \leq m-1,1 \leq \ell \leq m-1,$ then

$$g_j^{(\ell)}(|x|) \leq C\sum_{j=1}^{\ell}|x|^{jn-\ell}\int_{\omega_n|x|^n}^{\omega_nR^n} f_j(1+2\epsilon)(1+2\epsilon)^{-j+\frac{m}{n}-1}d(1+2\epsilon)$$

for $x \in B_R(0).x \in B_R(0)$. Whereas for $\ell = m\ell = m$ we get

$$g_j^{(m)}(|x|) \leq C\left(f_j(\omega_n|x|^n) \right.$$

$$\left. + \sum_{j=1}^{m-1}|x|^{jn-m}\int_{\omega_n|x|^n}^{\omega_nR^n}\sum_j f_j(1+2\epsilon)(1+2\epsilon)^{-j+\frac{m}{n}-1}d(1+2\epsilon)\right)$$

for some constant $C > 0.C > 0$.

Moreover,

$$|g_j(|x|)| \leq C\int_{\omega_n|x|^n}^{\omega_nR^n}\sum_j f_j\ (1+2\epsilon)(1+2\epsilon)^{\frac{m}{n}-1}d(1+2\epsilon).$$

$$|g_j(|x|)| \leq C\int_{\omega_n|x|^n}^{\omega_nR^n}\sum_j f_j\ (1+2\epsilon)(1+2\epsilon)^{\frac{m}{n}-1}d(1+2\epsilon).$$

Altogether, there exist a constant $C = C(m,n)C = C(m,n)$ such that

$$\sum_{k=0}^{m}\left|\nabla^k u(x)\right| \le C\left(f_j(\omega_n|x|^n) + \sum_{k=1}^{m-1}|x|^{kn-m}\int_{\omega_n|x|^n}^{\omega_n R^n}\sum_j f_j(r)\, r^{-k+\frac{m}{n}-1}dr \right.$$
$$\left. + \int_{\omega_n|x|^n}^{\omega_n R^n}\sum_j f_j(r)\, r^{\frac{m}{n}-1}dr \right) \qquad (4.19)$$

for a.e. $x \in B_R(0), x \in B_R(0)$,

whereas $\sum_{k=0}^{m}\left|\nabla^k u(x)\right| = 0 \sum_{k=0}^{m}\left|\nabla^k u(x)\right| = 0$ if $x \notin B_R(0)$. $x \notin B_R(0)$. It is easily verified that, for each $k = 1, \dots, m, k = 1, \dots, m$, the operators

$$f_j(1+2\epsilon) \mapsto (1+2\epsilon)^{k-\frac{m}{n}}\int_{1+2\epsilon}^{1}\sum_j f_j(r) r^{-k+\frac{m}{n}-1}dr \text{ and } f_j(1+2\epsilon)$$
$$\mapsto \int_{1+2\epsilon}^{1}\sum_j f_j(r) r^{\frac{m}{n}-1}dr$$

are bounded both in $L^1(0,1) L^1(0,1)$ and in $L^\infty(0,1), L^\infty(0,1)$, with norm depending only on $n\, n$ and mm. Hence, by an interpolation theorem by Calderón ([4] Theorem 2.12, Chapter 3), they are bounded in any rearrangement-invariant space, with norms depending only on its norms in $L^1(0,1) L^1(0,1)$ and in $L^\infty(0,1)$. $L^\infty(0,1)$. Therefore, owing to boundedness of the dilation operator in rearrangement-invariant space, we infer from (4.19) that

$$\sum_{k=0}^{m}\left\|\nabla^k u\right\|_{X(\lambda)} \le C \left\|\sum_j f_j(\omega_n R^n r)\right\|_{X(0,1)} \qquad (4.20)$$

for some positive constant $C = C(m, n, R, |\lambda|). C = C(m, n, R, |\lambda|)$. On the other hand, one has that

$$(Tr\,u)(x') = \chi_{(0,R)}(|x'|)\int_{\omega_n|x'|^n}^{\omega_n R^n}\int_{r_1}^{\omega_n R^n}\cdots\int_{r_{m-1}}^{\omega_n R^n}\sum_j f_j(r_m)r_m^{-m+\frac{m}{n}}\,dr_m\ldots dr_1$$

$$= \chi_{(0,R)}(|x'|)\int_{k(\omega_d|x'|^d)^{\frac{n}{d}}}^{\omega_n R^n}\int_{r_1}^{\omega_n R^n}\cdots\int_{r_{m-1}}^{\omega_n R^n}\sum_j f_j(r_m)r_m^{-m+\frac{m}{n}}\,dr_m\ldots dr_1$$

$$= \chi_{(0,R)}(|x'|)\int_{k(\omega_d|x'|^d)^{\frac{n}{d}}}^{\omega_n R^n}\int_{r_1}^{\omega_n R^n}\cdots\int_{r_{m-1}}^{\omega_n R^n}\sum_j f_j(r_m)r_m^{-m+\frac{m}{n}}\,dr_m\ldots dr_1$$

for $x' \in \lambda_d, x' \in \lambda_d$,

where $k = \frac{\omega_n}{(\omega_d)^{\frac{n}{d}}} k = \frac{\omega_n}{(\omega_d)^{\frac{n}{d}}}$ .Via this formula, one can deduce, via an argument as in the proof of ([14] Inequality 3.41) , that there exists a constant $C = C(\lambda) C = C(\lambda)$ such that

$$\|Tr\,u\|_{Y(\lambda_d)} \geq C\left\|\int_{(1+2\epsilon)^{\frac{n}{d}}}^{1}\sum_j f_j(\omega_n R^n r)r^{-1+\frac{m}{n}}\,dr\right\|_{Y(0,1)} \qquad (4.21)$$

$$\|Tr\,u\|_{Y(\lambda_d)} \geq C\left\|\int_{(1+2\epsilon)^{\frac{n}{d}}}^{1}\sum_j f_j(\omega_n R^n r)r^{-1+\frac{m}{n}}\,dr\right\|_{Y(0,1)} \qquad (4.21)$$

.

Inequality (1.12) follows from (4.20) and (4.21), owing to the arbitrariness of $f_j f_j$. In the case when $m \geq n, m \geq n$, inequality (1.12) holds for every pair of rearrangement-invariant spaces $X(0,1)\,X(0,1)$ and $Y(0,1)\,Y(0,1)$. Indeed,

$$\left\|\int_{(1+2\epsilon)^{\frac{n}{d}}}^{1}\sum_j f_j(r)r^{-1+\frac{m}{n}}\,dr\right\|_{L^\infty(0,1)} = \int_0^1\sum_j f_j(r)r^{-1+\frac{m}{n}}\,dr \leq \sum_j \|f_j\|_{L^1(0,1)},$$

for every $f_j \in \mathcal{M}_+(0,1), f_j \in \mathcal{M}_+(0,1)$, and hence the assertion follows from (2.6).

Proof of Theorem 1.1 Since $-1+\frac{m}{n}+\frac{d}{n} \geq 0, -1+\frac{m}{n}+\frac{d}{n} \geq 0$, condition (3.3) holds for $\epsilon = -\frac{d}{n} + 1\epsilon = -\frac{d}{n} + 1$ and $\omega(1-2\epsilon) = (1-2\epsilon)^{-1+\frac{m}{n}}.\omega(1-2\epsilon) = (1-2\epsilon)^{-1+\frac{m}{n}}$. Thus, by Proposition 3.1,

the functional $\|\cdot\|_{(X_{d,n}^{m})'(0,1)}\|\cdot\|_{(X_{d,n}^{m})'(0,1)}$ is a rearrangement-invariant function norm ,and

$$\left\| \int_{(1+2\epsilon)^{\frac{n}{d}}}^{1} \sum_{j} f_j \,(1+2\epsilon)(1+2\epsilon)^{-1+\frac{m}{n}}\, d(1+2\epsilon) \right\|_{X_{d,n}^{m}(0,1)} \leq \sum_{j} \|f_j\|_{X(0,1)} \qquad (4.22)$$

for every $f_j \in \mathcal{M}_{+}(0,1).f_j \in \mathcal{M}_{+}(0,1)$. Moreover,$\|\cdot\|_{X_{d,n}^{m}(0,1)}$ $\|\cdot\|_{X_{d,n}^{m}(0,1)}$is the optimal rearrangement-invariant function norm which renders (4.22) true. Hence, in particular, by Theorem 3.1, the trace embedding (1.6) holds [19].

In order to prove the optimality of the target norm, suppose that $\|\cdot\|_{Y(0,1)}\|\cdot\|_{Y(0,1)}$ is any rearrangement-invariant function norm such that

$Tr: W^{m}X(\lambda) \to Y(\lambda_d)$.

Then, by Theorem 3.1, inequality (1.12) holds for some positive constant $C_2 C_2$ and for all $f_j \in \mathcal{M}_{+}(0,1).f_j \in \mathcal{M}_{+}(0,1)$.Thus, by the optimality of the rearrangement-invariant function norm$\|\cdot\|_{X_{d,n}^{m}(0,1)}$ $\|\cdot\|_{X_{d,n}^{m}(0,1)}$ in (4.22), we necessarily have that $X_{d,n}^{m}(0,1) \to Y(0,1)$. $X_{d,n}^{m}(0,1) \to Y(0,1)$.

This proves the optimality of the same rearrangement-invariant function norm in (1.6). The proof is complete.

Proof of Corollary 1.2. The equivalence of (1.7) and (1.8) is a straightforward consequence of the optimality of the function norm $\|\cdot\|_{X^m_{d,n}(0,1)}\|\cdot\|_{X^m_{d,n}(0,1)}$ and of the first embedding in (2.6). As for the equivalence of (1.7) and (1.9), observe that, by Theorem 1.3, embedding (1.7) holds if and only if

$$\sup_{f_j \in \mathcal{M}_+(0,1)\,\|\sum_j f_j\|_{X(0,1)} \le 1} \left\| \int_{(1-2\epsilon)^{\frac{n}{d}}}^{1} \sum_j f_j(1+2\epsilon)(1+2\epsilon)^{-1+\frac{m}{n}}\, d(1+2\epsilon) \right\|_{L^\infty(0,1)} < \infty,$$

namely if and only if

$$\sup_{f_j \in \mathcal{M}_+(0,1)\|\sum_j f_j\|_{X(0,1)} \le 1} \int_0^1 \sum_j f_j(1+2\epsilon)(1+2\epsilon)^{-1+\frac{m}{n}}\, d(1+2\epsilon) < \infty.$$

$$\sup_{f_j \in \mathcal{M}_+(0,1)\|\sum_j f_j\|_{X(0,1)} \le 1} \int_0^1 \sum_j f_j(1+2\epsilon)(1+2\epsilon)^{-1+\frac{m}{n}}\, d(1+2\epsilon) < \infty.$$

Since the latter supremum agrees with $\left\|(1+2\epsilon)^{-1+\frac{m}{n}}\right\|_{X'(0,1)}$, $\left\|(1+2\epsilon)^{-1+\frac{m}{n}}\right\|_{X'(0,1)}$, the equivalence of (1.7) and (1.9) follows.

Proof of Theorem 1.4. We have that

$$\left\|\sum_j g_j\right\|_{\left((x^k_{\ell,n})^h_{d,\ell}\right)'(0,1)} = \left\|(1-2\epsilon)^{-1+\frac{h}{\ell}} \int_0^{(1-2\epsilon)^{\frac{d}{\ell}}} \sum_j g_j^*(1+2\epsilon)\, d(1+2\epsilon)\right\|_{(x^k_{\ell,n})'(0,1)}$$

$$= \left\|(1-2\epsilon)^{-1+\frac{k}{n}} \int_0^{(1-2\epsilon)^{\frac{d}{\ell}}} \left[\tau^{-1+\frac{h}{\ell}} \int_0^{\tau^{\frac{d}{\ell}}} \sum_j g_j^*(r)\, dr\right]^* (1+2\epsilon)\, d(1+2\epsilon)\right\|_{X'(0,1)} \quad (4.23)$$

for $g_j \in \mathcal{M}_+(0,1) g_j \in \mathcal{M}_+(0,1)$. Suppose first that $k+h \geq n$. $k+h \geq n$. Then, by Coroallary 1.2,

$$X_{\ell,n}^{k+h}(0,1) = L^\infty(0,1). \quad (4.24)$$

On the other hand,

$$\left\| \sum_j g_j \right\|_{\left((X_{\ell,n}^k)_{d,\ell}^h\right)'(0,1)} \leq \sum_j \|g_j\|_{L^1(0,1)} \left\| (1-2\epsilon)^{-1+\frac{k}{n}} \int_0^{(1-2\epsilon)^{\frac{\ell}{n}}} \left[\tau^{-1+\frac{h}{\ell}} \chi_{(0,1)}(\tau)\right]^* (1+2\epsilon) d(1+2\epsilon) \right\|_{X'(0,1)}$$

for every $g_j \in \mathcal{M}_+(0,1) g_j \in \mathcal{M}_+(0,1)$. If $h < l$, $h < l$, then $\left[\tau^{-1+\frac{h}{\ell}} \chi_{(0,1)}(\tau)\right]^* (1+2\epsilon) = (1+2\epsilon)^{-1+\frac{h}{\ell}}$, $\left[\tau^{-1+\frac{h}{\ell}} \chi_{(0,1)}(\tau)\right]^* (1+2\epsilon) = (1+2\epsilon)^{-1+\frac{h}{\ell}}$, and, consequently,

$$\left\| (1-2\epsilon)^{-1+\frac{k}{n}} \int_0^{(1-2\epsilon)^{\frac{\ell}{n}}} \left[\tau^{-1+\frac{h}{\ell}} \chi_{(0,1)}(\tau)\right]^* (1+2\epsilon) d(1+2\epsilon) \right\|_{X'(0,1)}$$

$$= \frac{\ell}{h} \left\| (1-2\epsilon)^{-1+\frac{k}{n}+\frac{h}{n}} \right\|_{X'(0,1)} \leq \frac{\ell}{h} \|\chi_{(0,1)}\|_{X'(0,1)},$$

since $k+h \geq n k+h \geq n$. If $h < l, h < l$, then $\left[\tau^{-1+\frac{h}{\ell}} \chi_{(0,1)}(\tau)\right]^* \leq 1, \left[\tau^{-1+\frac{h}{\ell}} \chi_{(0,1)}(\tau)\right]^* \leq 1$, and hence

$$\left\| (1-2\epsilon)^{-1+\frac{k}{n}} \int_0^{(1-2\epsilon)^{\frac{\ell}{n}}} \left[\tau^{-1+\frac{h}{\ell}} \chi_{(0,1)}(\tau) \right]^* (1+2\epsilon) d(1+2\epsilon) \right\|_{X'(0,1)}$$

$$\leq \left\| (1-2\epsilon)^{-1+\frac{k}{n}+\frac{\ell}{n}} \right\|_{X'(0,1)} \leq \| \chi_{(0,1)} \|_{X'(0,1),}$$

since $k+\ell \geq nk+\ell \geq n$ by our assumptions. Consequently,

$$\left\| \sum_j g_j \right\|_{\left((X^k_{\ell,n})^h_{d,\ell} \right)'(0,1)} \leq \max\left\{ 1, \frac{\ell}{n} \right\} \sum_j \| g_j \|_{L^1(0,1),}$$

for $g_j \in \mathcal{M}_+(0,1) g_j \in \mathcal{M}_+(0,1),$ whence $L^1(0,1) = \left((X^k_{\ell,n})^h_{d,\ell} \right)'(0,1), L^1(0,1) = \left((X^k_{\ell,n})^h_{d,\ell} \right)'(0,1),$ up to equivalent norms. Thus, by (2.3) and (2.7),

$$(X^k_{\ell,n})^h_{d,\ell}(0,1) = L^\infty(0,1). \qquad (4.25)$$

Coupling (4.25) with (4.24), implies that $(X^k_{\ell,n})^h_{d,\ell}(0,1) = X^{k+h}_{\ell,n}(0,1). (X^k_{\ell,n})^h_{d,\ell}(0,1) = X^{k+h}_{\ell,n}(0,1).$

Suppose next that $k+h<nk+h<n$. Then the assumption $k+\ell \geq nk+\ell \geq n$ entails that $h<\ell\, h<\ell$.

We have that

$$\left\| \sum_j g_j \right\|_{(X^{k+h}_{d,n})'(0,1)}$$

$$= \left\| (1-2\epsilon)^{-1+\frac{k+h}{n}} \int_0^{(1-2\epsilon)^{\frac{d}{n}}} \sum_j g_j^* (1+2\epsilon) d(1+2\epsilon) \right\|_{X'(0,1)} . \qquad (4.26)$$

On setting $\alpha = \frac{k}{n}, \beta = \frac{\ell}{n}, \gamma = \frac{h}{\ell}\alpha = \frac{k}{n}, \beta = \frac{\ell}{n}, \gamma = \frac{h}{\ell}$ and $\delta = \frac{d}{\ell}, \delta = \frac{d}{\ell}$, all the requirements in (3.14) are satisfied, since $k + h < nk + h < n$ entails that $\alpha + \beta\gamma < 1, k + \ell \geq n$ $\alpha + \beta\gamma < 1, k + \ell \geq n$ implies that $\alpha + \beta \geq 1$, $\alpha + \beta \geq 1$, and $\ell - d \leq h\ell - d \leq h$ implies that $\gamma + \delta \geq 1. \gamma + \delta \geq 1$. Therefore, by (4.23),(4.26) and Theorem 3.4 there exists a positive constant CC such that

$$\|\Sigma_j g_j\|_{\left(\left(X^k_{\ell,n}\right)^h_{d,\ell}\right)'(0,1)} \leq C \Sigma_j \|g_j\|_{\left(X^{k+h}_{d,n}\right)'(0,1)}$$

$$\|\Sigma_j g_j\|_{\left(\left(X^k_{\ell,n}\right)^h_{d,\ell}\right)'(0,1)} \leq C \Sigma_j \|g_j\|_{\left(X^{k+h}_{d,n}\right)'(0,1)}$$ for $g_j \in \mathcal{M}_+(0,1)$, $g_j \in \mathcal{M}_+(0,1)$, namely,

$$\left(X^{k+h}_{d,n}\right)'(0,1) \to \left(\left(X^k_{\ell,n}\right)^h_{d,\ell}\right)'(0,1). \qquad (4.27)$$

Conversely, by (2.2) and the monotonicity of the functio

$$1 + 2\epsilon \mapsto g_j^{**}\left((1 + 2\epsilon)^{\frac{d}{\ell}}\right) 1 + 2\epsilon \mapsto g_j^{**}\left((1 + 2\epsilon)^{\frac{d}{\ell}}\right)$$

$$\left\|\sum_j g_j\right\|_{\left(\left(X^k_{\ell,n}\right)^h_{d,\ell}\right)'(0,1)}$$

$$= \left\|(1-2\epsilon)^{-1+\frac{k}{n}}\int_0^{(1-2\epsilon)^{\frac{\ell}{n}}}\left[\tau^{-1+\frac{h}{\ell}}\int_0^{\tau^{\frac{d}{\ell}}}\sum_j g_j^*(r)dr\right]^*(1+2\epsilon)d(1+2\epsilon)\right\|_{X'(0,1)}$$

$$\geq \left\|(1-2\epsilon)^{-1+\frac{k}{n}}\int_0^{(1-2\epsilon)^{\frac{\ell}{n}}} s^{-1+\frac{h}{\ell}}\int_0^{s^{\frac{d}{\ell}}}\sum_j g_j^*(r)dr\,ds\right\|_{X'(0,1)}$$

$$= \left\|(1-2\epsilon)^{-1+\frac{k}{n}}\int_0^{(1-2\epsilon)^{\frac{\ell}{n}}}\sum_j g_j^{**}\left(s^{\frac{d}{\ell}}\right)s^{-1+\frac{h+d}{\ell}}\,ds\right\|_{X'(0,1)}$$

$$\geq \left\|(1-2\epsilon)^{-1+\frac{k}{n}}\sum_j g_j^{**}\left((1-2\epsilon)^{\frac{d}{n}}\right)\int_0^{(1-2\epsilon)^{\frac{\ell}{n}}} s^{-1+\frac{h+d}{\ell}}\,ds\right\|_{X'(0,1)}$$

$$= \frac{\ell}{h+d}\left\|(1-2\epsilon)^{-1+\frac{k+h+d}{n}}\sum_j g_j^{**}\left((1-2\epsilon)^{\frac{d}{n}}\right)\right\|_{X'(0,1)}$$

$$= \frac{\ell}{h+d}\left\|\sum_j g_j\right\|_{\left(X^{h+k}_{d,n}\right)'(0,1),}$$

for $g_i \in \mathcal{M}_+(0,1), g_i \in \mathcal{M}_+(0,1),$ whence

$$\left(\left(X_{\ell,n}^{k}\right)_{d,\ell}^{h}\right)'(0,1) \to \left(X_{d,n}^{h+k}\right)'(0,1). \qquad (4.28)$$

Coupling (4.27) and (4.28) tells us that $\left(\left(X_{\ell,n}^{k}\right)_{d,\ell}^{h}\right)'(0,1) = \left(X_{d,n}^{h+k}\right)'(0,1)$, $\left(\left(X_{\ell,n}^{k}\right)_{d,\ell}^{h}\right)'(0,1) = \left(X_{d,n}^{h+k}\right)'(0,1)$, or equivalently, $\left(\left(X_{\ell,n}^{k}\right)_{d,\ell}^{h}\right)(0,1) = \left(X_{d,n}^{h+k}\right)(0,1)$ $\left(\left(X_{\ell,n}^{k}\right)_{d,\ell}^{h}\right)(0,1) = \left(X_{d,n}^{h+k}\right)(0,1)$ whence (1.13) follows.

5. Trace embeddings for Lorentz-Sobolev and Orlicz-Sobolev spaces

we collect some new trace embeddings,with optimal targets, for custoumary Sobolev type spaces built upon Lorentz or Orlicz spaces.They follow from an application of our general results, via suitable Hardy type inequalities. We begin with Lorentz-Sobolev spaces.

Theorem 5.1.[Optimal trace embeddings in Lorentz-Sobolev spaces.] Let n, d, mn, d, m, and $\lambda\lambda$ be as in Theorem 1.1. Assume that either $0 < \epsilon < \infty 0 < \epsilon < \infty$ and $0 \geq \epsilon \geq -\infty, 0 \geq \epsilon \geq -\infty$, or $\epsilon = 0\epsilon = 0$ or $\epsilon = \infty, \epsilon = \infty$. Then

$$Tr: W^m L^{(1+\epsilon,1+\epsilon)}(\lambda) \to \begin{cases} L^{\frac{1+\epsilon d}{n-m(1+\epsilon)},1+\epsilon}(\lambda_d) & \text{if } m < n \text{ and } -\frac{1}{2} \leq \epsilon < \frac{1}{2}, \\ L^{\infty,1+\epsilon;-1}(\lambda_d) & \text{if } m < n\ , \epsilon = \frac{-3}{2} \text{ or } \epsilon > 0, \\ L^{\infty}(\lambda_d) & \text{otherwise.} \end{cases} \qquad (5.1)$$

Moreover, the target spaces in (5.1) are optimal among all rearrangement-invariant spaces on $\lambda_d.\lambda_d$.

Proof. Set $X(0,1) = L^{(1+\epsilon,1+\epsilon)}(0,1) X(0,1) = L^{(1+\epsilon,1+\epsilon)}(0,1)$.

Then, by (5.1), the denfinition of Lorentz spaces and (2.7),

$$\left\|\sum_j g_j\right\|_{(X^m_{d,n})'(0,1)} = \left\|(1-2\epsilon)^{\frac{m+d-n}{n}} \sum_j g_j^{**}\left(\tau^{\frac{d}{n}}\right)\right\|_{L^{(1+\epsilon)'},(1+\epsilon)'(0,1)} = \left\|(1-2\epsilon)^{\left(\frac{1}{(1+\epsilon)'}-\frac{1}{(1+\epsilon)'}\right)}\left[\tau^{\frac{m+d-n}{n}}\sum_j g_j^{**}\left(\tau^{\frac{d}{n}}\right)\right]^*\right\|_{L^{(1+\epsilon)'}(0,1)}$$

$$\left\|\sum_j g_j\right\|_{(X^m_{d,n})'(0,1)} = \left\|(1-2\epsilon)^{\frac{m+d-n}{n}} \sum_j g_j^{**}\left(\tau^{\frac{d}{n}}\right)\right\|_{L^{(1+\epsilon)'},(1+\epsilon)'(0,1)} = \left\|(1-2\epsilon)^{\left(\frac{1}{(1+\epsilon)'}-\frac{1}{(1+\epsilon)'}\right)}\left[\tau^{\frac{m+d-n}{n}}\sum_j g_j^{**}\left(\tau^{\frac{d}{n}}\right)\right]^*\right\|_{L^{(1+\epsilon)'}(0,1)}$$

,

every$g_j \in \mathcal{M}_+(0,1)$$g_j \in \mathcal{M}_+(0,1)$. Assume first that $m < n$ $m < n$, and either $-\frac{1}{2} \le \epsilon < \frac{1}{2}$$-\frac{1}{2} \le \epsilon < \frac{1}{2}$, or $\epsilon = 0, \epsilon = \frac{-3}{2}$ $\epsilon = 0, \epsilon = \frac{-3}{2}$ and $\epsilon > 0$$\epsilon > 0$. Fix any $g_j \in \mathcal{M}_+(0,1)$. $g_j \in \mathcal{M}_+(0,1)$.

Trivially,

$$\left\|\sum_j g_j\right\|_{(X^m_{d,n})'(0,1)} \le \left\|(1-2\epsilon)^{\left(\frac{1}{(1+\epsilon)'}-\frac{1}{(1-\epsilon)'}\right)} \sup_{1-2\epsilon \lesssim 1+2\epsilon \lesssim 1} (1+2\epsilon)^{\frac{m+d-n}{n}} \sum_j g_j^{**}\left((1+2\epsilon)^{\frac{d}{n}}\right)\right\|_{L^{(1+\epsilon)'}(0,1)}.$$

$$\left\|\sum_j g_j\right\|_{(X^m_{d,n})'(0,1)} \le \left\|(1-2\epsilon)^{\left(\frac{1}{(1+\epsilon)'}-\frac{1}{(1-\epsilon)'}\right)} \sup_{1-2\epsilon \lesssim 1+2\epsilon \lesssim 1} (1+2\epsilon)^{\frac{m+d-n}{n}} \sum_j g_j^{**}\left((1+2\epsilon)^{\frac{d}{n}}\right)\right\|_{L^{(1+\epsilon)'}(0,1)}.$$

Now, there exists a positive constant $C = C(1+\epsilon, 1-\epsilon, m, d, n)$ $C = C(1+\epsilon, 1-\epsilon, m, d, n)$ such that

$$\left\|(1-2\epsilon)^{\left(\frac{1}{(1+\epsilon)'}-\frac{1}{(1-\epsilon)'}\right)} \sup_{1-2\epsilon\leq 1+2\epsilon\leq 1} (1+2\epsilon)^{\frac{m+d-n}{n}} \sum_j g_j^{**}\left((1+2\epsilon)^{\frac{d}{n}}\right)\right\|_{L^{1+\epsilon'}(0,1)}$$

$$\leq C \left\|(1-2\epsilon)^{\left(\frac{1}{(1+\epsilon)'}-\frac{1}{(1+\epsilon)'}\right)+\frac{m+d-n}{n}} \sum_j g_j^{**}\left((1-2\epsilon)^{\frac{d}{n}}\right)\right\|_{L^{(1+\epsilon)'}(0,1)} . (5.2)$$

If $\epsilon > 0\epsilon > 0$ (and hence necessarily $\epsilon > 0\epsilon > 0$), then (5.2) follows from

([23] Theorem 3.2).When $\epsilon = 0,\epsilon = 0$, inequality (5.2) is a consequence of equality

$$\sup_{\frac{1}{2}<\epsilon\leq 0}\left((1-2\epsilon)^{\frac{1}{(1+\epsilon)'}} \sup_{1-\epsilon\leq 1+2\epsilon\leq 1}(1+2\epsilon)^{\frac{m+d-n}{n}}\sum_j g_j^{**}\left((1+2\epsilon)^{\frac{d}{n}}\right)\right)$$

$$= \sup_{0<1+2\epsilon\leq 1}(1+2\epsilon)^{\frac{m+d-n}{n}+\frac{1}{(1+\epsilon)'}}\sum_j g_j^{**}\left((1+2\epsilon)^{\frac{d}{n}}\right).$$

We have therefore shown that

$$\left\|\sum_j g_j\right\|_{(X_{d,n}^m)'(0,1)}$$

$$\leq C\left\|(1-2\epsilon)^{\left(\frac{1}{(1+\epsilon)'}-\frac{1}{(1+\epsilon)'}\right)+\frac{m+d-n}{n}}\sum_j g_j^{**}\left((1-2\epsilon)^{\frac{d}{n}}\right)\right\|_{L^{(1+\epsilon)'}(0,1)} \quad (5.3)$$

for $g_j \in \mathcal{M}_+(0,1) g_j \in \mathcal{M}_+(0,1)$. Conversely, one trivially has that

$$\left\| (1-2\epsilon)^{\left(\frac{1}{(1+\epsilon)'}-\frac{1}{(1+\epsilon)'}\right)+\frac{m+d-n}{n}} \sum_j g_j^{**}\left((1-2\epsilon)^{\frac{d}{n}}\right) \right\|_{L^{(1+\epsilon)'}(0,1)}$$

$$\leq \left\| (1-2\epsilon)^{\left(\frac{1}{(1+\epsilon)'}-\frac{1}{(1+\epsilon)'}\right)} \sup_{1-2\epsilon\leq 1+2\epsilon\leq 1} (1+2\epsilon)^{\frac{m+d-n}{n}} \sum_j g_j^{**}\left((1+2\epsilon)^{\frac{d}{n}}\right) \right\|_{L^{(1+\epsilon)'}(0,1)} \quad (5.4)$$

for every $g_j \in \mathcal{M}_+(0,1)$. $g_j \in \mathcal{M}_+(0,1)$. Observe that the function $\varrho(1-2\epsilon) = (1-2\epsilon)^{\frac{m+d-n}{n}}$ $\varrho(1-2\epsilon) = (1-2\epsilon)^{\frac{m+d-n}{n}}$ satisfies (3.9). Hence, by Lemma 3.2,

$$\left\| (1-2\epsilon)^{\left(\frac{1}{(1+\epsilon)'}-\frac{1}{(1+\epsilon)'}\right)} \sup_{1-2\epsilon\leq 1+2\epsilon\leq 1} (1+2\epsilon)^{\frac{m+d-n}{n}} \sum_j g_j^{**}\left((1+2\epsilon)^{\frac{d}{n}}\right) \right\|_{L^{(1+\epsilon)'}(0,1)}$$

$$\leq C \left\| (1-2\epsilon)^{\left(\frac{1}{(1+\epsilon)'}-\frac{1}{(1+\epsilon)'}\right)} \left[\tau^{\frac{m+d-n}{n}}\left(\tau^{\frac{d}{n}}\right)\right]^{**}(1-2\epsilon) \right\|_{L^{(1+\epsilon)'}(0,1),} \quad (5.5)$$

for some constant CC, and for every $g_j \in \mathcal{M}_+(0,1)$. $g_j \in \mathcal{M}_+(0,1)$. We next claim that there exists a positive constant $C = C(1+\epsilon, 1-\epsilon, m, d, n)$, $C = C(1+\epsilon, 1-\epsilon, m, d, n)$, such that

$$\left\|(1-2\epsilon)^{\left(\frac{1}{(1+\epsilon)'}-\frac{1}{(1+\epsilon)'}\right)}\left[\tau^{\frac{m+d-n}{n}}\sum_j g_j^{**}\left(\tau^{\frac{d}{n}}\right)\right]^{**}(1-2\epsilon)\right\|_{L^{(1+\epsilon)'}(0,1),} \leq C\left\|(1-2\epsilon)^{\left(\frac{1}{(1+\epsilon)'}-\frac{1}{(1+\epsilon)'}\right)}\left[\tau^{\frac{m+d-n}{n}}\sum_j g_j^{**}\left(\tau^{\frac{d}{n}}\right)\right]^{*}(1-2\epsilon)\right\|_{L^{(1+\epsilon)'}(0,1),}$$

$$\left\|(1-2\epsilon)^{\left(\frac{1}{(1+\epsilon)'}-\frac{1}{(1+\epsilon)'}\right)}\left[\tau^{\frac{m+d-n}{n}}\sum_j g_j^{**}\left(\tau^{\frac{d}{n}}\right)\right]^{**}(1-2\epsilon)\right\|_{L^{(1+\epsilon)'}(0,1),} \leq C\left\|(1-2\epsilon)^{\left(\frac{1}{(1+\epsilon)'}-\frac{1}{(1+\epsilon)'}\right)}\left[\tau^{\frac{m+d-n}{n}}\sum_j g_j^{**}\left(\tau^{\frac{d}{n}}\right)\right]^{*}(1-2\epsilon)\right\|_{L^{(1+\epsilon)'}(0,1),}$$

(5.6)

for $g_j \in \mathcal{M}_+(0,1)$ $g_j \in \mathcal{M}_+(0,1)$. If $\epsilon > 0$, $\epsilon > 0$, inequality (5.6) follows from ([3] Theorem 1.7). If $\epsilon = 0$, $\epsilon = 0$, for each $\epsilon \geq 0$ $\epsilon \geq 0$ and $h \in \mathcal{M}_+(0,1)$, $h \in \mathcal{M}_+(0,1)$, one has that

$$\sup_{\frac{1}{2}<\epsilon\leq 0}(1-2\epsilon)^{\frac{1}{(1+\epsilon)'}}h^{**}(1-2\epsilon)$$

$$= \sup_{\frac{1}{2}<\epsilon\leq 0}(1-2\epsilon)^{\frac{1}{(1+\epsilon)'}-1}\int_0^{1-2\epsilon}(1+2\epsilon)^{-\frac{1}{(1+\epsilon)'}}h^*(1+2\epsilon)(1+2\epsilon)^{\frac{1}{(1+\epsilon)'}}d(1+2\epsilon)$$

$$\leq \sup_{\frac{1}{2}<\epsilon\leq 0}(1-2\epsilon)^{\frac{1}{(1+\epsilon)'}}h^*(1-2\epsilon)\sup_{\frac{1}{2}<\epsilon\leq 0}(1-2\epsilon)^{\frac{1}{(1+\epsilon)'}-1}\int_0^{1-2\epsilon}(1+2\epsilon)^{-\frac{1}{(1+\epsilon)'}}d(1+2\epsilon),$$

whence

$$\sup_{\frac{1}{2}<\epsilon\leq 0} (1-2\epsilon)^{\frac{1}{(1+\epsilon)'}} h^{**}(1-2\epsilon) \leq (1+\epsilon) \sup_{\frac{1}{2}<\epsilon\leq 0} (1-2\epsilon)^{\frac{1}{(1+\epsilon)'}} h^{*}(1-2\epsilon).$$

$$\sup_{\frac{1}{2}<\epsilon\leq 0} (1-2\epsilon)^{\frac{1}{(1+\epsilon)'}} h^{**}(1-2\epsilon) \leq (1+\epsilon) \sup_{\frac{1}{2}<\epsilon\leq 0} (1-2\epsilon)^{\frac{1}{(1+\epsilon)'}} h^{*}(1-2\epsilon).$$

As a consequence, (5.6) holds also in this case. Combining (5.4) , (5.5) and (5.6) tells us that

$$\left\| (1-2\epsilon)^{\frac{1}{(1+\epsilon)'}-\frac{1}{(1+\epsilon)'}+\frac{m+d-n}{n}} \sum_j g_j^{**}\left((1-2\epsilon)^{\frac{d}{n}}\right) \right\|_{L^{(1+\epsilon)'}(0,1)} \leq C \sum_j \|f_j\|_{(X^m_{d,n})'(0,1)} \quad (5.7)$$

for some constant $C = C(1+\epsilon, 1+\epsilon, m, d, n)$ $C = C(1+\epsilon, 1+\epsilon, m, d, n)$, and for every $g_j \in \mathcal{M}_+(0,1)$. $g_j \in \mathcal{M}_+(0,1)$.

Set $r = \frac{d}{n}$ $r = \frac{d}{n}$, and observe that

$$\left\| (1-2\epsilon)^{\frac{1}{(1+\epsilon)'}-\frac{1}{(1+\epsilon)'}+\frac{m+d-n}{n}} \sum_j g_j^{**}\left((1-2\epsilon)^{\frac{d}{n}}\right) \right\|_{L^{(1+\epsilon)'}(0,1)} = \left(\frac{n}{d}\right)^{\frac{1}{(1+\epsilon)'}} \left\| \sum_j g_j \right\|_{L^{(r,(1+\epsilon)')}(0,1)} \quad (5.8)$$

for every $g_j \in \mathcal{M}_+(0,1)$ $g_j \in \mathcal{M}_+(0,1)$. From (5.3), (5.7) and (5.8) we infer that

$$(X^m_{d,n})'(0,1) = L^{(r,(1+\epsilon)')}(0,1),$$ $$(X^m_{d,n})'(0,1) = L^{(r,(1+\epsilon)')}(0,1),$$

up to equivalent norms. Now, if $\epsilon = 0, \epsilon < \frac{-3}{2}$, $\epsilon = 0, \epsilon < \frac{-3}{2}$, then $r > 1, r > 1$, and therefore, by (2.12), $(X^m_{d,n})'(0,1) = L^{r,(1+\epsilon)'}(0,1).\ (X^m_{d,n})'(0,1) = L^{r,(1+\epsilon)'}(0,1).$

Hence, by (2.7),

$$X^m_{d,n}(0,1) = L^{\frac{1+\epsilon d}{n-m(1+\epsilon)'},1+\epsilon}(0,1) X^m_{d,n}(0,1) = L^{\frac{1+\epsilon d}{n-m(1+\epsilon)'},1+\epsilon}(0,1).$$

If $\epsilon = 0 \text{ or } \epsilon = \frac{-3}{2}, \epsilon = 0 \text{ or } \epsilon = \frac{-3}{2}$, and $\epsilon > 0, \epsilon > 0$, then $r = 1\, r = 1$, and, by (2.13), $X^m_{d,n}(0,1) = L^{\infty,1+\epsilon;-1}(0,1)$. $X^m_{d,n}(0,1) = L^{\infty,1+\epsilon;-1}(0,1)$.It remains to consider the cases when either $m \geq n m \geq n$ or $\epsilon > 0, \epsilon > \frac{-3}{2} \epsilon > 0, \epsilon > \frac{-3}{2}$,or $\epsilon = \frac{-3}{2}\ \epsilon = \frac{-3}{2}$and $\epsilon = 0. \epsilon = 0$. In each of these cases,one has that $\left\| (1-2\epsilon)^{-1+\frac{m}{n}} \right\|_{L^{(1+\epsilon)',(1+\epsilon)'}(0,1)} < \infty$, $\left\| (1-2\epsilon)^{-1+\frac{m}{n}} \right\|_{L^{(1+\epsilon)',(1+\epsilon)'}(0,1)} < \infty$, whence $X^m_{d,n}(0,1) = L^\infty(0,1), X^m_{d,n}(0,1) = L^\infty(0,1)$, by Corollary 1.2.The proof is complete. Let us now focus on trace embedding for Orlicz-Sobolev spaces. Let $n, m, d \in \mathbb{N} n, m, d \in \mathbb{N}$ be as in the statement of Theorem1.1.Given a Young function $A\,A$, denote by $L^A(\lambda) L^A(\lambda)$the corresponding Orlicz space. Let $m < n m < n$. We may assume, without loss of generality, that

$$\int_0 \left(\frac{1-2\epsilon}{A(1-2\epsilon)} \right)^{\frac{m}{n-m}} d(1-2\epsilon) < \infty. \qquad (5.9)$$

Indeed, AA can be replaced, if necessary, by a Young function equivalent near infinity, which renders (5.9) true, such replacement leaving the Orlicz-Sobolev space $W^m L^A(\lambda)\, W^m L^A(\lambda)$ unchanged (up to equivalent norms). If $m < n, m < n$, and the integral

$$\int^{\infty} \left(\frac{1-2\epsilon}{A(1-2\epsilon)}\right)^{\frac{m}{n-m}} d(1-2\epsilon) \qquad (5.10)$$

diverges, define the function $H_m: [0,\infty) \to [0,\infty)$ $H_m: [0,\infty) \to [0,\infty)$ as

$$H_m(1+2\epsilon) = \left(\int_0^{1+2\epsilon} \left(\frac{1-2\epsilon}{A(1-2\epsilon)}\right)^{\frac{m}{n-m}} d(1-2\epsilon)\right)^{\frac{n-m}{n}}$$

$$H_m(1+2\epsilon) = \left(\int_0^{1+2\epsilon} \left(\frac{1-2\epsilon}{A(1-2\epsilon)}\right)^{\frac{m}{n-m}} d(1-2\epsilon)\right)^{\frac{n-m}{n}}$$

for $\epsilon \geq \frac{-1}{2}, \epsilon \geq \frac{-1}{2}$, and the Young function $A_{m,d} A_{m,d}$ by

$$A_{m,d}(1-2\epsilon) = \int_0^{H_m^{-1}(1-2\epsilon)} \left(\frac{A(1+2\epsilon)}{1+2\epsilon}\right)^{\frac{d-m}{n-m}} H_m(1+2\epsilon)^{\frac{d-m}{n-m}}\, d(1+2\epsilon)$$

$$A_{m,d}(1-2\epsilon) = \int_0^{H_m^{-1}(1-2\epsilon)} \left(\frac{A(1+2\epsilon)}{1+2\epsilon}\right)^{\frac{d-m}{n-m}} H_m(1+2\epsilon)^{\frac{d-m}{n-m}}\, d(1+2\epsilon)$$

for$\epsilon \geq \frac{1}{2} \epsilon \geq \frac{1}{2}$.

The following result provides us with an optimal Orlicz target in Orlicz-Sobolev trace embeddings. It follows from Theorem 1.3, via ([11] Theorem 3.5).

Theorem 5.2. [Orlicz-Sobolev trace embeddings with an optimal Orlicz target] Let $n, d, m\; n, d, m$ and $\lambda\lambda$ be as in Theorem 1.1. Let AA be a Young function fulfilling (5.9).

Then (5.11)

$$Tr: W^m L^A(\lambda) \to \begin{cases} L^{A_{m,d}}(\lambda_d) \text{ if } m < n, \text{and the integral (5.10) diverges,} \\ L^{\infty}(\lambda_d) \text{ if either } m \geq n, \text{or } m < n \text{ and the integral (5.10) converges.} \end{cases} \quad (5.11)$$

Moreover, the target spaces in (5.11) are optimal among all Orlicz spaces.

In the exambles below, we present applications of Theorem 5.2 to couple of customary instances of Orlicz-Sobolev spaces. In what follows, the notation $\Phi(L)(\lambda)\Phi(L)(\lambda)$ is used to denote the Orlicz space on $\lambda\lambda$ built upon a Young function equivalent to the function Φ Φ near infinity.The notations $\Phi(L)(\lambda_d)$ $\Phi(L)(\lambda_d)$ and $W^m\Phi(L)(\lambda)W^m\Phi(L)(\lambda)$ are used accordingly.

Example 5.3. Assume that either $\epsilon > 0\epsilon > 0$ and $\alpha \in \mathbb{R},\alpha \in \mathbb{R}$, or $\epsilon = 0\epsilon = 0$ and $\alpha \geq 0.\alpha \geq 0$. An application of Theorem 5.2 yields

$$Tr: W^m L^{(1+\epsilon)}(\log L)^{\alpha}(\lambda) \to \begin{cases} L^{\frac{(1+\epsilon)d}{n-m(1+\epsilon)}}(\log L)^{\frac{\alpha d}{n-m(1+\epsilon)}}(\lambda_d) \quad \text{if} \quad -\frac{1}{2} \leq \epsilon < \frac{1}{2} \\ \exp L^{\frac{n}{n-m-\alpha m}}(\lambda_d) \quad \text{if} \quad \epsilon = 0 \text{ or } \epsilon = \frac{-3}{2} \text{ and } \alpha < \frac{n-m}{m} \\ \exp\exp L^{\frac{n}{n-m}}(\lambda_d) \quad \text{if} \quad \epsilon = 0 \text{ or } \epsilon = \frac{-3}{2} \text{ and } \alpha = \frac{n-m}{m} \\ L^{\infty}(\lambda_d) \text{ if either } \epsilon = 0 \text{ or } \epsilon = \frac{-3}{2} \text{ and } \alpha > \frac{n-m}{m}, \text{or } \epsilon > 0 \text{ or } \epsilon > \frac{-3}{2} \\ \text{if either } \epsilon = 0 \text{ or } \epsilon = \frac{-3}{2} \text{ and } \alpha > \frac{n-m}{m}, \text{or } \epsilon > 0 \text{ or } \epsilon > \frac{-3}{2} \end{cases}$$

all the range spaces being optimal in the class of Orlicz spaces.

Example 5.4. Assume that $(1+\epsilon)(1+\epsilon)$ and $\alpha\alpha$ are as in Example 5.3.Then, one can obtain from Theorem 5.2 that

$$Tr: W^m L^{1+\epsilon}(\log\log L)^{\alpha}(\lambda) \to \begin{cases} L^{\frac{1+\epsilon d}{n-m(1+\epsilon)}}(\log\log L)^{\frac{\alpha d}{n-m(1+\epsilon)}} & \text{if } -\frac{1}{2} \le \epsilon < \frac{1}{2}, \\ \exp\left(L^{\frac{n}{n-m}}(\log L)^{\frac{\alpha m}{n-m}}\right)(\lambda_d) & \text{if } \epsilon = 0 \text{ or } \epsilon = \frac{-3}{2}, \\ L^{\infty}(\lambda_d) & \text{if } \epsilon > 0 \text{ or } \epsilon > \frac{-3}{2}. \end{cases}$$

Moreover, the range spaces are sharp in the framework of Orlicz spaces on $\lambda_d.\lambda_d$.

Although the target space in the first embedding in Theorem 5.2 is optimal in the framework of Orlicz spaces, it can be improved if the class of admissible target is enlarged to include all rearrangement-invariant spaces. It turns out that the optimal rearrangement-invariant target space isn the first case of (5.11) is an Orlicz-Lorentz space. Assume that $m < nm < n$ and AA is a Young function that makes the integral in (5.10) diverge. Let aa be the left-continuous derivative of AA, so that aa and AA are related as in (2.14). Define the Young function $E_m\, E_m$ as

$$E_m((1-2\epsilon)) = \int_0^{1-2\epsilon} e_m(1+2\epsilon)\,d(1+2\epsilon)$$

$$E_m((1-2\epsilon)) = \int_0^{1-2\epsilon} e_m(1+2\epsilon)\,d(1+2\epsilon) \quad \text{for } \epsilon \ge \frac{1}{2}, \epsilon \ge \frac{1}{2},$$

where $e_m\, e_m$ is the non-decreasing, left-continuous function in $[0,\infty)[0,\infty)$ obeying

$$e_m^{-1}(1+2\epsilon) = \left(\int_{a^{-1}(1+2\epsilon)}^{\infty}\left(\int_0^{\tau}\left(\frac{1}{a(1-2\epsilon)}\right)^{\frac{m}{n-m}} d(1-2\epsilon)\right)^{-\frac{1}{1+2\epsilon}} \frac{d\tau}{a(\tau)^{\frac{n}{n-m}}}\right)^{\frac{m}{n-m}}$$

$$e_m^{-1}(1+2\epsilon) = \left(\int_{a^{-1}(1+2\epsilon)}^{\infty} \left(\int_0^{\tau} \left(\frac{1}{a(1-2\epsilon)} \right)^{\frac{m}{n-m}} d(1-2\epsilon) \right)^{-\frac{1}{1+2\epsilon}} \frac{d\tau}{a(\tau)^{\frac{n}{n-m}}} \right)^{\frac{m}{n-m}}$$

for $\epsilon \geq \frac{-1}{2}.\epsilon \geq \frac{-1}{2}$.

Finally,let $L\left(\frac{1}{1+2\epsilon},\frac{n}{d},E_m\right)(\lambda_d)L\left(\frac{1}{1+2\epsilon},\frac{n}{d},E_m\right)(\lambda_d)$ be the Orlicz-Lorentz space associated with the function norm defined as in (2.15).

Theorem 5.5. [Orlicz-Sobolev trace embeddings with optimal rearrangement-invariant target] Let n, d, mn, d, m and $\lambda\lambda$ be as in Theorem 1.1. Let AA be a Young function fulfilling (5.9). Assume that $m < nm < n$, and the integral in (5.10) diverges.Then

$$Tr: W^m L^A(\lambda) \to L\left(\frac{1}{1+2\epsilon},\frac{n}{d},E_m\right)(\lambda_d), \qquad (5.12)$$

and the target space in (5.12) is optimal among all rearrangement-invariant spaces.Embedding (5.12) follows from Theorem 1.3, via an analogous argument as in the proof of [11].

Example 5.6. Let $(1+\epsilon)(1+\epsilon)$and $\alpha\alpha$ be as in Example 5.3. From Theorem 5.5 one can deduce that

$$Tr: W^m L^{1+\epsilon}(\log L)^{\alpha}(\lambda) \to \begin{cases} L^{\frac{(1+\epsilon)d}{n-m(1+\epsilon)},1+\epsilon;\frac{\alpha}{1+\epsilon}}(\lambda_d) & \text{if } -\frac{1}{2} \leq \epsilon < \frac{1}{2} \\ L^{\infty,\frac{n}{m};\frac{m\alpha}{n}-1}(\lambda_d) & \text{if } \epsilon = 0, \epsilon = \frac{-3}{2} \text{ and } \alpha < \frac{n-m}{m} \\ L^{\infty,\frac{1}{1+2\epsilon};-\frac{m}{n}-1}(\lambda_d) & \text{if } \epsilon = 0, \epsilon = \frac{-3}{2} \text{ and } \alpha = \frac{n-m}{m} \end{cases}$$

up to equivalent norms.Furthermore, all the range spaces are optimal among rearrangement-invariant spaces [19].

Conclusion:

Finally optimal target spaces are exhibited in arbitrary-order Sobolev type embeddings for traces of n-dimensional functions on lower dimensional subspaces. Sobolev spaces built upon any rearrangement-invariant norm are allowed. A key step in our approach consists in showing that any trace embedding can be reduced to a one-dimensional inequality for a Hardy type operator depending only on n and on the dimension of the relevant subspace. This can be regarded as an analogue for trace embeddings of a well-known symmetrization principle for first-order Sobolev embeddings for compactly supported functions. The stability of the optimal target space under iterations of Sobolev trace embeddings is also established, and is part of the proof of our reduction principle. A a consequence, we derive new trace embeddings, with improved (optimal) target spaces, for classical Sobolev, Lorentz-Sobolev and Orlicz-Sobolev spaces.

References:

(1)D.R. Adams, Traces of potentials II, Indiana Univ.Math.J.22 (1973), 907-918.

(2)R. A. Adams, Sobolev Spaces, Pure and Applied Mathematics Vol. 65, Academic Press, Orlando 1975.

(3)M. Arińo and B.Muckenhoupt, Maximal functions on classical Lorentz spaces and Hardy’s inequality with weights for non-increasing functions, Trans. Amer.Math.Soc. 320 (1990), 727-735.

(4)C.Bennett and R. Sharpley, Interpolation of Operators, Pure and Applied Mathematics Vol. 129, Academic Press, Boston 1988.

(5)[5] H. Brézis and S.Wainger, A note on limiting cases of Sobolev embeddings and convolution inequalities, Comm. Partial Diff. Eq.5 (1980), 773-789.

(6)J. Brothers, W.Ziemer, Minimal rearrangements of Sobolev functions, J. Reine. A ngew. Math.384 (1988), 153-179.

(7)V.I. Burenkov, Sobolev spaces on domains, Teubner, Stuttgart, 1998.

(8)P.Cavaliere and A. Cianchi, Classical and approximate Taylor expansions of weakly differentiable functions, Ann. Acad. Scient.Fen.,to appear.

(9)A. Cianchi, Symmetrization and second-order Sobolev inequalities, Ann. Mat. Pura Appl. 183 (2004), 45-77.

(10)[10]A. Cianchi, Optimal Orlicz-Sobolev embeddings, Rev. Mat. Iberoamericana (2004),427-474.

(11)A. Cianchi, Orlicz-Sobolev boundary trace embeddings, Math. Zeit. 266 (2010), 431-449.

(12)A. Cianchi, D.E Edmunds and P. Gurka, On weightec Poincaré inequalities, Math. Nachr. 180 (1996), 15-41.

(13)A. Cianchi and A. Ferone, On symmetric functional of the gradient having symmetric equidistributed minimizers, SIAM J.Math.Anal. 38 (2006), 279-308.

(14)A. Cianchi, R.Kerman and L.Pick, Boundary trace inequalities and rearrangements, J. Anal. Math 105 (2008), 241-265.

(15)[15] A. Cianchi and L.Pick, Sobolev embeddings into BMO,VMO and L_∞,L_∞, Ark. Mat. 36 (1998), 317-340.

(16)A. Cianchi and L.Pick,An optimal endpoint trace embeddings, Anal. Inst.Fourier 60, 3 (2010), 939-951.

(17)A. Cianchi ,L.Pick and L. Slavíková, Higher-order Sobolev embeddings and isoperimetric inequalities, Adv.Math.,appear.

(18)A. Cianchi and M. Randolfi, On the modulus of continuity of weakly differentiable functions, Indiana Univ.Math. J.60 (2011), 1939-1973.

(19)A. Cianchi and L. Pick, Optimal Sobolev trace embeddings, Trans. Amer. Math. Soc.(2016), http.//dx dol.org/10.1090/tran/6606. In press, article electronically published on January 19, 2016.

(20)W.D.Evans, B.Opic and L.Pick, Interpolation of integral operators on scales of generalized Lorentz-Zygmund spaces, Math. Nachr. 182 (1996),127-181.

(21)E. Gagliardo, Proprietá di alcune classi di funzioni di piú variabili, Ric.Mat. 7 (1958), 102-137.

(22)F.Gazzola, H.-C. Grunau and G. Sweers, Optimal Sobolev and Hardy-Rellich constants under Navier boundary conditions, Ann. Mat. Pura Appl. 189 (2010),475-486.

(23)A. Gogatishvili, B.Opic and L. Pick, Weighted inequalities for Hardy-type operators involving suprema, Collect. Math. 57,3 (2006), 227-255.

(24)K. Hansson, Imbedding theorems of Sobolev type in potential theory, Math. Scand. 45 (1979), 77-102.

(25) R. Kerman, L.Pick, Optimal Sobolev imbeddings, Forum Math.18,4 (2006), 535-570.

(26) V. G. Maz'ya, Sobolev spaces with applications to elliptic partial differential equations, Springer Ser. Sov Math. Springer, Berlin, (2011).

(27)L.Nirenberg, On elliptic partial differential equations, Ann. Sc. Norm.Sup. Pisa 13 (1959), 115-162.

(28)R.O'Neil, Convolution operators and $L(p,q)L(p,q)$ spaces, Duke Math. J. 30 (1963), 129-142.

(29)B.Opic and L.Pick, On generalized Lorentz-Zygmund spaces, Math. Ineq. Appl. 2 (1999), 391-467.

(30)[30] J. Peetre, Espaces d'interpolation et théoréme de Soboleff, Ann. Inst. Fourier 16 (1966), 279-317.

(31) L. Pick, A. Kufner, O. John and S. Fučík, Function Spaces, Volume 1, 2nd Revised and Extended Edition, De Gruyter Series in Nonlinear Analysis and applications 14, De Gruyter, Berlin (2013).

(32)S .I. Pohozaev, On the imbedding Sobolev theorem for $pl = n$ $pl = n$, Doklady Conference, Section Math. Moscow Power Inst. (1965), 158-170 (Russian).

(33)[33] E. Sawyer, Boundedness of classical operators on classical Lorentz spaces, Studia Math. 96 (1990), 145-158.

(34)S.L. Sobolev, Applications of functional Analysis in Mathematical Physics, Transl. of Mathem. Monographs. American Math. Soc., Providence, R.1. 7, (1963).

(35)E.Sten, Singular integrals and differentiability properties of functions, Princeton University Press, Princeton, New Jersey, (1970).

(36)G. Talenti, Best constant in Sobolev inequality, Ann. Mat. Pura App. 110 (1976), 353-372.

(37)C. Tarsi, Adams' inequality and limiting Sobolev embeddings into Zygmund spaces, Potentional Anal. 37 (2012), 353-385.

(38)N.S. Trudinger, On imbeddings into Orlicz spaces and some applications, J. Mech. Anal. 17 (1967), 473-483.

(39)V.I. Yudovich, Some estimates connected with integral operators and with solutions of elliptic equations, Soviet Math. Dokl. 2 (1961), 746-749 (Russian).

(40)W.P. Ziemer, Weakly differentiable functions, Graduate texts in Math. 120, Springer, Berlin, (1989).