Common Fixed Point for Weakly Compatible and Sequentially Continuous Mappings in Metric Spaces

النقطة الثابتة العامة لأجل التوافق الضعيف والرواسم المستمرة المتتابعة في الفضاءات المترية

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Abstract

This study aims the show important tools, possible applications, existence of fixed point theory, results for mappings and contractive mappings on complete G-Metric Spaces. The common fixed point theorem for maps satisfying a general contractive condition of integral type for a pair of weakly compatible mappings in fuzzy metric spaces, using property in cone metric spaces, of expansive mappings in generalized metric spaces, in complex valued metric spaces and for weakly compatible mappings under contractive conditions of integral type in complex valued metric spaces are considered. The existence of multiple solutions and result of one nontrivial solution of the boundary value problems for nonlinear second. Order differential equations and for some nonlinear systems with singular laplacian and one nontrivial solution for two point are obtained.

Key Words:

Common Fixed Point / Weakly Compatible/ Sequentially Continuous/ Mappings / Metric Spaces

المستخلص

تهدف هذه الدراسة إلى إظهار الأدوات المهمة، والتطبيقات الممكنة، ووجود نظرية النقطة الثابتة، ونتائج التعيينات والتعيينات الانقباضية على مساحات G-Metric كاملة. نظرية النقطة الثابتة الشائعة للخرائط التي تفي بشرط انكماش عام من النوع المتكامل لزوج من التعيينات المتوافقة بشكل ضعيف في المساحات المترية الضبابية، باستخدام خاصية في المساحات المترية المخروطية، والتعيينات الموسعة في المساحات المترية المعممة، وفي المساحات المترية ذات القيمة المعقدة وللضعيف. تؤخذ في الاعتبار التعيينات المتوافقة في ظل الشروط التعاقدية للنوع المتكامل في المساحات المترية ذات القيمة المعقدة. وجود حلول متعددة ونتيجة حل واحد غير بديهي لمشاكل القيمة الحدية للثانية اللاخطية. ترتيب المعادلات التفاضلية وبالنسبة لبعض الأنظمة غير الخطية ذات حل لا بلاسي مفرد ومحلول واحد غير بديهي لنقطتين.

الكلمات المفتاحية:

النقطةالثابتة العامة / التوافق الضعيف /الرواسم المستمرة المتتابعة/ الفضاءات المترية

1.0 Introduction:

We deal with some important information of various kinds of weak commutativity for the computation of fixed points we start frome the concept of weakly commuting maps to the latest biased maps of type(Ar)and type (As)applications can be found in dynamic programming, approximation theory , variational inequalities and solution of nonlinear integral equations we show some fixed point results for mapping satisfying sufficient conditions on complete G-metric space, also we showed that if the G-metric space is symmetric, then the existence and uniqueness of these fixed point results follow from well-known theorems in usual metric space( ).

We show some common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces both in the sense of Kramosil and Michalek and in the sense of George and Veeramani by using the new property and give some examples. we show some common fixed point theorems for different types of contractive conditions.

We establish the precise condition concerning the behavior of at in8nity and zero for the existence of solutions with prescribed nodal properties. Then we derive the existence and the multiplicity of nodal solutions to the problem. Our argument is based on the shooting method together with the Strum's comparison theorem. We establish several results related to existence, nonexistence or bifurcation of positive solutions for the boundary value problem where $\Omega \subset RN$ ($N \geq 2$) is a smooth bounded domain, is a positive parameter, and f is smooth and has a sublinear growth. The main feature in the presence of the singular nonlinearity g combined with the convection term Multiple critical points theorems for non-differentiable functionals are established.

Two common fixed theorems for weakly compatible mappings satisfying general contractive conditions of integral type in metric spaces are show and an illustrative example is provided using (E.A) property and (CLR) property common fixed point results for weakly compatible mappings, satisfying integral type contractive condition in complex valued metric spaces are investigated.

With a homeomorphism of the ball are considered, under various boundary conditions on a compact interval For non- homogeneous Cauchy, terminal and some Sturm-Liouville boundary condi-tions including in particular the Dirichlet-Neumann and Neumann-Dirichlet conditions, existence of a solution is proved for arbitrary continuous right- hand sides f: For Neumann boundary conditions, some restrictions upon f are required, although, for Dirichlet boundary conditions, the restrictions are only upon _ and the boundary values. For periodic boundary conditions, both _ and f have to be suitably restricted. Existence results of positive solutions for a two point boundary value problem are established. No asymptotic condition on the nonlinear term either at zero or at infinity is required. The aim of this paper is to present a coincidence point theorem for sequentially weakly continuous maps. Moreover, as a consequence, a critical point theorem for functionals possibly containing a nonsmooth part is obtained.

1.**1 Objective of the Study**

This study aims to fulfill the following objective:

show important tools, possible applications, existence of fixed point theory, results for mappings and contractive mappings on complete G-Metric Spaces.

1.2 **The importance of the study:**

The importance of this study is the importance of the subject that is addressed in the study, which is the Common Fixed Point for Weakly Compatible and Sequentially Continuous

Mappings in Metric Spaces:

Introduce complex valued metric spaces and obtain sufficient conditions for the existence of common fixed points of a pair of mappings satisfying contractive type conditions

2.1 Preliminaries

In this section, we recall some definitions and useful results which are already in the literature.

Definition[1]: Let (X,d) be ametric space and $J=[0,l]$. A mapping $W: X\times X\times J\rightarrow X$ iscalled convex structure on X iffor all $x,y,u\in X$ and $\lambda\in J, d(u,W(x,y,\lambda))\leq\lambda d(u,x)+(1-\lambda)d(u,y)$.A metric spacewith convex structure iscalled convex metric space(2).

(a) Anonempty subset K of a convex metric space X issaidtobeconvex if $W(x,y,\lambda)\in K$ forall $x,y\in K$ and $\lambda\in[0,1]$,

b) K issaidtobe q-starshaped if there exists apoint $q\in K$ such that $W(x,y,\lambda)\in K$ forall $x\in K$ and $\lambda\in J$.

Definition[2]: Aconvex metric space X is said to satisfy the condition(*), if $d(W(x,p,\lambda),W(y,p,\lambda)) \le d(x,y)$ for all $x,y \in X$ and $\lambda \in [0,1]$.[3]

Definition [3]: see [1]. Let X be a nonempty set, and let $G: X \times X \times X \to \boldsymbol{R}^{+}$be a function satisfying the following properties:

$(G1) G(x,y,z) = 0$ if $x = y = z$;

$(G2) 0 < G(x,x,y)$; for all $x,y \in X$ with $x \neq y$;

$(G3) G(x,x,y) \le G(x,y,z)$ for all $x,y,z \in X$ with $z \neq y$;

$(G4) G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$, (symmetry in all three variables

$(G5) G(x,y,z) \le G(x,a,a) = G(a,y,z)$, for all $x,y,z,a \in X$,(rectangle inequality)

Then the function G is called ageneralized metric, or, more specifically, a G-metricon X, and the pair$(X,G)$is called aG-metric space[4].

Definition [4]: (see [1]).Let(X,G)be aG-metric space, and let(x_n)be sequence of points of , a point $x \in X$ is said to be the limit of the sequence(x_n), if $lim_{n,m\to\infty} G(x,x_n,x_m) = 0$, and one says that the sequence(x_n), is G-convergent to x.

Thus, that if $x_n \to x$ in aG-metric space(X,G), then for any $\epsilon > 0$, there exists $N \in N$ such that $G(x, x_n,x_m) < \epsilon$ for all $n,m \ge N$.

Definition [5]: (see [1]). Let(X,G)be aG-metric space, a sequence (x_n), is called G-Cauchy if for every $\epsilon > 0$, there is $N \in N$ such that $G(x_n, x_m, x_l) < \epsilon$, for all $n,m,l \ge N$; that is, if $G(x_n, x_m, x_l) \to 0$ as $n,m,l \to \infty$.

Definition [5]. Let$(X,G)$and(X',G') be two G-metric spaces, and let $f: (X,G) \to (X',G')$ be a function, then f is said to be G-continuous at a point $a \in X$ if and only if, given $\epsilon > 0$, there exists $\delta > 0$ such that $x,y \in X$; and $G(a,x,y) < \delta$ implies $G'(f(a),f(x),f(y)) < \epsilon$.

A function f is G-continuous at X if and only if it is G-continuous at all $a \in X$.

Definition [6]: A G-metric space $(X,G)$is called symmetric G-metric space if $G(x,y,y) = G(y,x,x)$for all $x,y \in X$.

Definition [7]:.A G-metric space (X,G) is said to be G-complete (or complete G-metric if every G-Cauchy sequence in $(X,G)$is G-convergent in (X,G).

Mapping on Complete G-Metric Spaces

During the sixties, the notion of 2-metric space introduced by Gähler as a generalization of usual notion of metric space(X, d). But different proved that there is no relation between these two functions, for instance, show that 2-metric need not be continuous function, further there is no easy relationship between results obtained in the two settings(5).

In 1992, BapureDhage in his Ph.D. thesis introduce a new class of generalized metric space calledD-metric spaces ([59,60]).In a subsequent series, Dhage attempted to develop topological structures in such spaces (see [60–62]). He claimed that D-metrics provide a generalization of ordinary metric functions and went on to present several fixed point results.

But in 2003 in collaboration with Brailey Sims, we demonstrated , that most of the claims concerning the fundamental topological structure of D-metric space are incorrect, so, we introduced more appropriate notion of generalized metric space as follows.

Complex Valued Metric Spaces

Since the appearance of the Banach contraction mapping principle, a number of articles have been dedicated to the improvement and generalization of that result. Most of these deal with the generalizations of the contractive condition in metric spaces.

Ghaler(6) generalized the idea of metric space and introduced a 2-metric space which was followed by a number dealing with this generalized space. Plenty of material is also available in other generalized metric spaces, such as, rectangular metric spaces, semi metric spaces, pseudo metric spaces, probabilistic metric spaces, fuzzy metric spaces, Quasi metric spaces, Quasi semi metric spaces, D-metric spaces, and cone metric spaces (7).

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z2), \mathrm{Im}(z_1) \leq \mathrm{Im}(z_2)$

It follows that

$$z_1 \precsim z_2$$

if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2), \quad \mathrm{Im}(z_1) < \mathrm{Im}(z_2),$

(ii) $Re(z_1) < Re(z_2), \quad \mathrm{Im}(z_1) = \mathrm{Im}(z_2),$

(iii) $Re(z_1) < Re(z_2), \quad \mathrm{Im}(z_1) < \mathrm{Im}(z_2),$

(iv) $Re(z_1) = Re(z_2), \quad \mathrm{Im}(z_1) = \mathrm{Im}(z_2)$

In particular, we will write $z_1 \lneqq z_2$ if $z_1 = z_2$ and one of (i), (ii), and (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied. Note that

$$0 \precsim z_1 \Rightarrow |z_1| < |z_2|,$$

$$z_1 \precsim z_2, z_2 \Rightarrow z_1 \prec z_3.$$

Definition [8]: Let X be a nonempty set. Suppose that the mapping $d: X \times X \to \mathbb{C}$, satisfies[8]:

(i) $0 \precsim d(x,y)$, for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$for all $x, y \in X$;

(iii)$d(x,y) \precsim d(x,z) + d(z,y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X, and (X,d) is called a complex valued metric space. A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that

$$B(x,r) = \{y \in X: d(x,y) \prec r\} \subseteq A.$$

A point $x \in X$ is called a limit point of A whenever for every $0 \prec r \in \mathbb{C}$,

$$B(x,r) \cap (A \backslash X) \neq \phi$$

A is called open whenever each element of A is an interior point of A. A subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B. The family

$$F = \{B(x,r): x \in X, 0 \prec r\}$$

is a sub-basis for a Hausdorff topology τon X.

Let x_n be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$, with $0 \prec c$ there is $n_0 \in N$such that for all $n > n_0, d(x_n, x) \prec c$, then $\{x_n\}$is said to be convergent, $\{x_n\}$converges to x and x is the limit point of $\{x_n\}$. We denote this by $\lim_n x_n = x$, or $x_n \to x$, as $n \to \infty$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in N$ such that for all $n > n_0, d(x_n, x_{n+m}) \prec c$, then $\{x_n\}$ is called a Cauchy sequence in(X,d). If every Cauchy sequence is convergent in (X,d), then $(X,d)$is called a complete complex valued metric space.

3.1 Main Results:

Theorem: Let (X,G) be a complete G-metric space, and let $T: X \to X$ be a mapping satisfying one of the following conditions[9] :

$$G(T(x),T(z)) \leq$$

$$|aG(x,y,z)+bG(x,T(x))+cG(y,T(y))+dG(z,T(z),T(z))| \qquad (1)$$

or

$$G(T(x),T(y),T(z)) \leq$$

$$|aG(x,y,z)+bG(x,x,T(x))+cG(y,y,T(y))+dG(z,z,T(z))| \qquad (2)$$

for all $x,y,z \in X$ where $0 \leq a+b+c+d < 1$, then T has a unique fixed point (sayu, i.e., $Tu = u$), and T is G-continuous at u.

Proof: Suppose that T satisfies condition (38), then for all $x,y \in X$, we have

$$G(Tx,Ty,Ty) \leq aG(x,y,y)+bG(x,Tx,Tx)+(c+d)G(y,Ty,Ty)$$

$$G(Ty,Tx,Tx) \leq aG(y,x,x)+bG(y,Ty,Ty)+(c+d)G(x,Tx,Tx) \qquad (3)$$

Suppose that (X,G) is symmetric, then by definition of metric (X,d_G) and (4) , we get

$$d_G(Tx,Ty) \leq ad_G(x,y)+\frac{c+d+b}{2}d_G(x,Tx)$$

$$+ad_G(x,y)+\frac{c+d+b}{2}d_G(y,Ty),\ \forall x,y \in X \qquad (4)$$

In this line, since $0 < a+b+c+d < 1$, then the existence and uniqueness of the fixed point follows from well-known theorem in metric spaceX, dGsee10.

However, if (X,G) is not symmetric then by definition of metric $(X,d_G)$and (3) , we get

$$d_G(Tx,Ty) \leq ad_G(x,y)+\frac{2(c+d+b)}{3}d_G(x,Tx)+\frac{2(c+d+b)}{3}d_G(y,Ty), \qquad (5)$$

for all $x,y \in X$, then the metric condition gives no information about this map since $0 < a+2(c+d+b)/3+2(c+d+b)/3$ need not be less than 1. But this can be proved by G-metric.

Let $x_0 \in X$ be an arbitrary point, and define the sequence (x_n) by $x_n = T^n x_0$. By (2), we have

$$G(x_n,x_{n+1},x_{n+1}) \leq aG(x_{n-1},x_n,x_n)bG(x_{n-1},x_n,x_n)$$

$$+(c+d)G(x_n,x_{n+1},x_{n+1}) \qquad (6)$$

then

$$G(x_n,x_{n+1},x_{n+1}) \leq \frac{a+b}{1-(c+d)} G(x_{n-1},x_n,x_n) \qquad (7)$$

Let $q=(a+b)/(1-(c+d))$, then $0 \leq q < 1$ since $0 \leq a+b+c+d < 1$.

So,

$$G(x_n,x_{n+1},x_{n+1}) \leq qG(x_{n-1},x_n,x_n) \qquad (8)$$

Continuing in the same argument, we will get

$$G(x_n,x_{n+1},x_{n+1}) \leq q^2 G(x_0,x_1,x_1) \qquad (9)$$

Moreover, for all $n,m \in N;\ n < m$, we have by rectangle inequality that

$$G(x_n,x_m,x_m) \leq G(x_n,x_{n+1},x_{n+1}) + G(x_{n+1},x_{n+2},x_{n+2}) +$$

$$G(x_{n+2},x_{n+3},x_{n+3}) + \cdots + G(x_{m-1},x_m,x_m)$$

$$\leq (q^n + q^{n+1} + \cdots + q^{m-1})G(x_0,x_1,x_1)$$

$$\leq \frac{q^n}{1-q} G(x_0,x_1,x_1) \qquad (10)$$

and so $\lim G(x_n,x_m,x_m)=0$, as $n,m \to \infty$. Thus (x_n) is G-Cauchy sequence. Due to the completeness of (X,G), there exists $u \in X$ such that (x_n) is G-converge to u.

Suppose that $T(u) \neq u$, then

$$G(x_n,T(u),T(u)) \leq aG(x_{n-1},u,u) + bG(x_{n-1},x_n,x_n)$$

$$+(c+d)G(u,T(u),T(u)) \qquad (11)$$

taking the limit as $n \to \infty$, and using the fact that the function G is continuous, then $G(u,T(u),T(u)) \leq (c+d)G(u,T(u),T(u))$. This contradiction implies that $u=T(u)$.

To prove uniqueness, suppose that $u \neq v$ such that $T(v)=v$, then

$$G(u,v,v) \leq aG(u,v,v) + bG(u,T(u),T(u))$$

$$+(c+d)G(v,T(v),T(v)) = aG(u,v,v) \qquad (12)$$

which implies that $u=v$.

To show that T is G-continuous at u, let $(y_n) \subseteq X$ be a sequence such that $\lim(y_n)=u$.

we can deduce that

$$G(u,T(y_n),T(y_n)) \leq aG(u,y_n,y_n)+bG(u,T(u),T(u))$$

$$+(c+d)G(y_n,T(y_n),T(y_n)) = aG(u,y_n,y_n)$$

$$+(c+d)G(y_n,T(y_n),T(y_n)) \quad (13)$$

and since $G(y_n,T(y_n),T(y_n)) \leq G(y_n,u,u)+G(u,T(y_n),T(y_n))$, we have that $G(u,T(y_n),T(y_n)) \leq \left(a/(1-(c+d))\right)G(u,y_n,y_n)+\left((c+d)/(1-(c+d))\right)G(y_n,u,u)$

Taking the limit as $n \to \infty$, from which we see that $G(u,T(y_n),T(y_n)) \to 0$ and so, by Proposition (1.2.7), $T(y_n) \to u = Tu$. It is proved that T is G-continuous at u.

If T satisfies condition (5), then the argument is similar to that above. However, to show that the sequence (x_n) is G-Cauchy, we start with

$$G(x_n,x_n,x_{n+1}) \leq aG(x_{n-1},x_{n-1},x_n)$$

$$+(b+c)G(x_{n-1},x_{n-1},x_n)+dG(x_n,x_n,x_{n+1}) \quad (14)$$

then

$$G(x_n,x_n,x_{n+1}) \leq \frac{a+b+c}{1-d}G(x_{n-1},x_{n-1},x_n). \quad (15)$$

Let $q=(a+b+c)/(1-d)$, then $0 \leq q < 1$ since $0 \leq a+b+c+d < 1$.

Continuing in the same way, we find that

$$G(x_n,x_n,x_{n+1}) \leq q^n G(x_0,x_0,x_1) \quad (16)$$

Then for all $n,m \in N;\ n < m$, we have by repeated use of the rectangle inequality $G(x_n,x_n,x_m) \leq (q^n/(1-q))G(x_0,x_0,x_1)$.

Corollary: Let (X,G), be a complete G-metric space and let $T: X \to X$ be a mapping satisfying one of the following conditions:

$$G(T^m(x),T^m(y),T^m(z)) \leq \{aG(x,y,y)+bG(x,T^m(x),T^m(x))+cG(y,T^m(y),T^m(y))+dG(z,T^m(z),T^m(z))\} \quad (17)$$

or

$$G(T^m(x),T^m(y),T^m(z)) \leq \{aG(x,y,y) + bG(x,x,T^m(x)) + cG(y,y,T^m(y)) + dG(z,zT^m(z))\}$$

(18)

for all $x, y, z \in X$, where $0 \leq a+b+c+d < 1$. Then T has a unique fixed point (say u), and T^m is G-continuous at u.

Proof: From the previous theorem, we see that T^m has a unique fixed point (say u) ,that is, $T^m(u) = u$. But $T(u) = T(T^m(u)) = T^{m+1}(u) = T^m(T(u))$, so $T(u)$ is another fixed point for T^m and by uniqueness $Tu = u$.

Common Fixed Point Theorems

We introduce complex valued metric spaces and obtain sufficient conditions for the existence of common fixed points of a pair of mappings satisfying contractive type conditions The results obtained substantially extend and improve several previous results, particularly of Branciari Rhoades and of Vijayaraju et al. A nontrivial example with uncountably many points is also provided to support the results presented herein.

Lemma[10]: Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in . Then $\{x_n\}$ converges to x if and only if $|d(x_n,x)| \to 0$ as $n \to \infty$.

Proof: Suppose that $\{x_n\}$ converges to x. For a given real number $\epsilon > 0$, let

$$c = \frac{\epsilon}{\sqrt{2}} + i\frac{\epsilon}{\sqrt{2}}.$$

Then $0 \prec c \in \mathbb{C}$ and there is a natural number N, such that

$$d(x_n,x) \prec c \quad \text{for all } n > N.$$

Therefore,

$$|d(x_n,x)| < |c| = \epsilon \quad \text{for all } n > N.$$

It follows that

$$|d(x_n,x)| \to 0 \ as \ \to \infty.$$

Conversely, suppose that $|d(x_n,x)| \to 0$ asn→∞. Then given $c \in \mathbb{C}$ with $0 \prec c$, there exists a real number $\delta > 0$, such that for $z \in \mathbb{C}$

$$|z| < \delta \Rightarrow z \prec c.$$

For this δ, there is a natural number N such that

$$|d(x_n,x)| < \delta \quad \text{for all} \quad n > N.$$

This means that $d(x_n,x) \prec c$ for all $n > N$. Hence $\{x_n\}$ converges to x.

Lemma [11]**:** Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in . Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n,x_{n+m})| \to 0$ as $n \to \infty$.

Proof: Suppose that $\{x_n\}$ is a Cauchy sequence. For a given real number $\epsilon > 0$, let

$$c = \frac{\epsilon}{\sqrt{2}} + i\frac{\epsilon}{\sqrt{2}}.$$

Then $0 \prec c \in \mathbb{C}$ and there is a natural number N, such that:

$$d(x_n,x_{n+m}) \prec c = \epsilon \quad \text{for all} \quad n > N.$$

Therefore,

$$|d(x_n,x_{n+m})| \prec |c| = \epsilon \quad \text{for all} \quad n > N.$$

It follows that

$$|d(x_n,x_{n+m})| \to 0 \quad as\, n \to \infty.$$

Conversely, suppose that $|\, d(x_n,x_{n+m})| \to 0$ as $n \to \infty$. For given $c \in \mathbb{C}$ with $0 \prec c$, there exists a real number $\delta > 0$, such that for $z \in \mathbb{C}$

$$|z| < \delta \Rightarrow z \prec c.$$

For this δ, there is a natural number N such that:

$$|d(x_n,x_{n+m})| < \delta \quad \text{for all} \quad n > N.$$

That is $d(x_n,x_{n+m}) \prec c$ for all $n > N$ and so $\{x_n\}$ is a Cauchy sequence.

Theorem [12]**:** Let (X,d)be a complete complex valued metric space and let the mappings $S,T\colon X \to X$ satisfy:

$$d(Sx,Ty) \precsim \lambda(x,y) + \frac{\mu d(x,Sx)d(y,Ty)}{1+d(x,y)}$$

for all $x,y \in X$, where λ,μ are nonnegative reals with $\lambda+\mu < 1$. Then S,T have a unique common fixed point.

Proof: Let x_0 be an arbitrary point in X and define

$$x_{2k+1} = Sx_{2k}$$

$$x_{2k+2} = Tx_{2k+1}, \qquad k = 0,1,2 \ldots.$$

Then,

$$d(x_{2k+1}, x_{2k+2}) = d(Sx_{2k}, Tx_{2k+1})$$

$$\lesssim \lambda d(x_{2k}, x_{2k+1}) + \frac{\mu d(x_{2k+1}, Tx_{2k+1}) d(x_{2k}, Sx_{2k})}{1 + d(x_{2k}, x_{2k+1})}$$

$$\lesssim \lambda d(x_{2k}, x_{2k+1}) + \frac{\mu d(x_{2k+1}, Tx_{2k+2}) d(x_{2k}, x_{2k+1})}{1 + d(x_{2k}, x_{2k+1})}$$

$$\lesssim \lambda d(x_{2k}, x_{2k+1}) + \mu d(x_{2k+1}, x_{2k+2})$$

$$since \;\; d(x_{2k}, x_{2k+1}) \lesssim 1 + d(x_{2k}, x_{2k+1})$$

$$\lesssim \frac{\lambda}{1 - \mu} d(x_{2k}, x_{2k+1})$$

Similarly,

$$d(x_{2k+2}, x_{2k+3}) = d(Sx_{2k+2}, Tx_{2k+1})$$

$$\lesssim \lambda (x_{2k+2}, x_{2k+1}) + \frac{\mu d(x_{2k+1}, Tx_{2k+1}) d(x_{2k+2}, Sx_{2k+2})}{1 + d(x_{2k}, x_{2k+1})}$$

$$\lesssim \lambda (x_{2k+2}, x_{2k+1}) + \frac{\mu d(x_{2k+1}, x_{2k+2}) d(x_{2k+2}, x_{2k+3})}{1 + d(x_{2k}, x_{2k+1})}$$

$$\lesssim \lambda (x_{2k+2}, x_{2k+1}) + \mu d(x_{2k+2}, x_{2k+3})$$

$$\lesssim \frac{\lambda}{1 - \mu} (x_{2k+2}, x_{2k+1})$$

$$\precsim \left[\frac{h^n}{1-h}\right] d(x_0, x_1)$$

and so

$$|d(x_m, n)| \precsim \frac{h^n}{1-h} |d(x_0, x_1)| \to 0, \quad as\ m, n \to \infty.$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $u \in X$ such that $x_n \to u$. It follows that $u = Su$, otherwise $d(u, Su) = z > 0$ and we would then have

$$z \precsim d(u, x_{2k+2}) + d(x_{2k+2}, Su)$$

$$\precsim d(u, x_{2k+2}) + d(Tx_{2k+1}, Su)$$

$$\precsim d(u, x_{2k+2}) + \lambda d(x_{2k+1}, u) + \frac{\mu d(x_{2k+1}, Tx_{2k+1}) d(u, Su)}{1 + d(u, x_{2k+1})}$$

$$\precsim d(u, x_{2k+2}) + \lambda d(x_{2k+1}, u) + \frac{\mu d(x_{2k+1}, x_{2k+2}) z}{1 + d(u, x_{2k+1})}$$

This implies that

$$|z| \leq |d(u, x_{2k+2})| + \lambda |d(x_{2k+1}, u)| + \frac{|\mu d(x_{2k+1}, x_{2k+2})||z|}{|1 + d(u, x_{2k+1})|}.$$

That is $|z| = 0$, a contradiction and, hence, $u = Su$. It follows similarly that $u = Tu$.

We now show that S and T have unique common fixed point. For this, assume that u^* in X is a second common fixed point of S and T. Then

$$d(u, u^*) = d(Su, Tu^*)$$

$$\precsim \lambda d(u, u^*) + \frac{\mu d(u, Su) d(u^*, Tu^*)}{1 + d(u, u^*)}$$

$$\precsim \lambda d(u, u^*)$$

This implies that $u^* = u$, completing the proof of the theorem.

Corollary [13]: Let (X, d)be a complete complex valued metric space and let the mapping $T: X \to X$ satisfy:

$$d(Tx, Ty) \precsim \lambda d(x, y) + \frac{\mu d(x, Tx) d(y, Ty)}{1 + d(x, y)}$$

for all $x, y \in X$, where λ, μ are nonnegative reals with $\lambda + \mu < 1$. Then T has a unique fixed point.

Corollary [14]**:** Let (X, d)be a complete complex valued metric space and $T: X \to X$ satisfy:

$$d(T^n x, T^n y) \precsim \lambda d(x, y) + \frac{\mu d(x, T^n x) d(y, T^n y)}{1 + d(x, y)}$$

for all $x, y \in X$, where λ, μ are nonnegative reals with $\lambda + \mu < 1$. Then T has a unique fixed point.

Proof: By Corollary (5.1.5) we obtain $v \in X$ such that

$$T^n v = v$$

The result then follows from the fact that

$$d(Tv, v) = d(TT^n v, T^n v) = d(TT^n v, T^n v)$$

$$\precsim \lambda d(Tv, v) + \frac{\mu d(Tv, T^n Tv) d(v, T^n v)}{1 + (Tv, v)}$$

$$\precsim \lambda d(Tv, v) + \frac{\mu d(Tv, TT^n v) d(v, v)}{1 + (Tv, v)} d(Tv, v)$$

3.1.4 Theorem [15]**:** Let $X = C([a, b], \mathbb{R}^n), a > 0$ and $d: X \times X \to \mathbb{C}$ is defined as follows: $d(x, y) = \max_{t \in [a,b]} \|x(t) - y(t)\|_\infty \sqrt{1 + a^2} e^{i \tan^{-1} a}$.

Consider the Urysohn integral equations

$$x(t) \int_a^b K_1(t, s, x(s)) ds + g(t), \qquad \text{(i)}$$

$$x(t) \int_a^b K_2(t, s, x(s)) ds + h(t), \qquad \text{(ii)}$$

where $t \in [a, b] \subset \mathbb{R}, x, g, h \in X$.

Suppose that $K_1, K_2: [a, b] \times [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ are such that $F_x, G_x \in X$ for each $x \in X$, where

$$F_x(t) \int_a^b K_1(t, s, x(s)) ds, \quad G_x(t), = \int_a^b K_2(t, s, x(s)) ds \text{ for all } t \in [a, b].$$

If there exist nonnegative reals λ , μ with $\lambda + \mu < 1$ such that for every $x, y \in X$

$$\left\|F_x(t) - G_y(t) + g(t) - h(t)\right\|_\infty \sqrt{1+a^2} e^{i\tan^{-1}a}$$

$$\precsim \lambda A(x,y)(t) + \mu B(x,y)(t)$$

for all $x, y \in X$, where

$$A(x,y)(t) = \|x(t) - y(t)\|_\infty \sqrt{1+a^2} e^{i\tan^{-1}a},$$

$$B(x,y)(t) = \frac{\|F_x(t) - g(t) - x(t)\|_\infty \left\|G_y(t) + h(t) - y(t)\right\|_\infty}{1 + d(x,y)} \sqrt{1+a^2} e^{i\tan^{-1}a},$$

then the system of integral equations (1) and (2) have a unique common solution.

Proof: Define $S, T: X \to X$ by

$$Sx = F_x + g, \qquad Tx = G_x + h.$$

Then

$$d(Sx, Ty) = \max_{t \in [a,b]} \left\|F_x(t) - G_y(t) + g(t)\right\|_\infty \sqrt{1+a^2} e^{i\tan^{-1}a},$$

$$d(x, Tx) = \max_{t \in [a,b]} \|F_x(t) - g(t)\|_\infty \sqrt{1+a^2} e^{i\tan^{-1}a}$$

and

$$d(y, Ty) = \max_{t \in [a,b]} \left\|G_y(t) + h(t) - y(t)\right\|_\infty \sqrt{1+a^2} e^{i\tan^{-1}a}.$$

It is easily seen that

$$d(Sx, Ty) \precsim \lambda d(x,y) + \frac{\mu d(x,Tx) d(y,Sy)}{1 + d(x,y)},$$

for every $x, y \in X$. By Theorem (5.1. 4), the Urysohn integral Eqs. (1) and (2) have a unique common solution.

Corollary: Let (X, d) be a complex valued metric space and let $\{x_n^r\}$ be a sequence in . Then $\{x_n^r\}$ converges to x if and only if $\sum |d(x_n^r, x^r)| \to 0$ as $n \to \infty$.

Corollary : Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in . Then $\{x_n^r\}$ is a Cauchy sequence if and only if $\sum |d(x_n^r, x_{n+m}^r)| \to 0$ as $n \to \infty$.

Proof: Suppose that $\{x_n^r\}$ is a Cauchy sequence. For a given real number $\epsilon > 0$, let

$$c = \frac{\epsilon}{\sqrt{2}} + i\frac{\epsilon}{\sqrt{2}}.$$

Then $0 \prec c \in \mathbb{C}$ and there is a natural number N, such that:

$$\sum d(x_n^r, x_{n+m}^r) \prec c = \epsilon \quad \text{for all } r, n > N.$$

Therefore,

$$\sum |d(x_n^r, x_{n+m}^r)| \prec |c| = \epsilon \quad \text{for all } r, n > N.$$

It follows that

$$\sum |d(x_n^r, x_{n+m}^r)| \to 0 \quad a_r s\, n \to \infty.$$

Conversely, suppose that $|d(x_n^r, x_{n+m}^r)| \to 0$ as $n \to \infty$. For given $c \in \mathbb{C}$ with $0 \prec c$, there exists a real number $\delta > 0$, such that for $z \in \mathbb{C}$

$$|z| < \delta \Longrightarrow z \prec c.$$

For this δ, there is a natural number N such that:

$$\sum |d(x_n^r, x_{n+m}^r)| < \delta \quad \text{for all } r, n > N.$$

That is $\sum d(x_n^r, x_{n+m}^r) \prec c$ for all $r, n > N$ and so $\{x_n^r\}$ is a Cauchy sequence.

Corollary : Let (X, d)be a complete complex valued metric space and $T: X \to X$ satisfy:

$$d(T^n x_m, T^n(x_m + \epsilon)) \precsim \lambda d(x, (x_m + \epsilon)) + \frac{\mu d(x_m, T^n x_m) d(y, T^n(x_m + \epsilon))}{1 + d(x_m, (x_m + \epsilon))}$$

for all $x, y \in X$, where λ, μ are nonnegative reals with $\lambda + \mu < 1$. Then T has a unique fixed point.

Proof: By Corollary (5.1.5) we obtain $v \in X$ such that

$$T^n v_m = v_m$$

The result then follows from the fact that

$$d(Tv_m, v_m) = d(TT^n v_m, T^n v_m) = d(TT^n v_m, T^n v_m)$$

$$\precsim \lambda d(Tv_m, v_m) + \frac{\mu d(Tv_m, T^n Tv_m) d(v_m, T^n v_m)}{1 + (Tv_m, v_m)}$$

$$\precsim \lambda d(Tv_m, v_m) + \frac{\mu d(Tv_m, TT^n v) d(v, v)}{1 + (Tv_m, v_m)} d(Tv_m, v_m)$$

Example: Let

$$x_1 = \{z \in \mathbb{C}: 0 \le Re(z) \le 1, \mathrm{Im}(z) = 0\},$$

$$x_2 = \{z \in \mathbb{C}: 0 \leq \text{Im}(z) \leq 1, Re(z) = 0\}$$

and let $X = X_1 \cup X_2$. Then with $z = x + iy$, define

$$Tz = \begin{cases} ix & if\ z \in x_1 \\ \frac{1}{2}y & if\ z \in x_2 \end{cases}$$

If d_u is usual metric on X then T is not contractive as

$$d_u(Tz_1, Tz_2) = |x_1 - x_2| = d_u(z_1, z_2) \qquad if\ z_1, z_2 \in X_1.$$

Therefore, the Banach contraction theorem is not valid to find the unique fixed point 0 of T. To apply the corollary, consider a complex valued

Metric $d: X \times X \to \mathbb{C}$as follows:

$$d(z_1, z_2) = \begin{cases} \frac{2}{3}|x_1 - x_2| + \frac{i}{2}|x_1 - x_2|, & \text{if} z_1, z_2 \in X_1 \\ \frac{1}{2}|y_1 - y_2| + \frac{i}{3}|y_1 - y_2|, & \text{if} z_1, z_2 \in X_1, \\ \left(\frac{2}{3}x_1 + \frac{1}{2}y_2\right) + i\left(\frac{1}{2}x_1 + \frac{1}{3}y_2\right), & \text{if} z_1, z_2 \in X_1, z_2 \in X_2, \\ \left(\frac{1}{2}y_1 + \frac{2}{3}y_2\right) + i\left(\frac{1}{3}y_1 + \frac{1}{2}y_2\right), & \text{if} z_1, z_2 \in X_2, z_2 \in X_1 \end{cases}$$

where $z_1 = x_1 + iy_1, z_1 = x_2 + iy_2 \in X$. Then (X, d) is a complete complex valued metric space and

$$d(Tz_1, Tz_2) = \tfrac{3}{4} d(z_1, z_2) \text{for all} z_1, z_2 \in X.$$

Example: Let $X = C([1,3], \mathbb{R}), a > 0$ and for every $x, y \in X$ let

$$M_{xy} = \max_{t \in [1,3]} |x(t) - y(t)|,$$

$$d(x, y) = M_{xy}\sqrt{1 + a^2}\, e^{i\tan^{-1}a}.$$

Define $T: X \to X$ by

$$T(x(t)) = 4 + \int_1^t (x(u) + u^2) e^{u-1} du, \qquad t \in [1,3]$$

For every $x, y \in X$

$$d(Tx, Ty) = M_{TxTy}\sqrt{1 + a^2}\, e^{i\tan^{-1}a} = \max_{t \in [1,3]} |Tx(t) - Ty(t)| \sqrt{1 + a^2}\, e^{i\tan^{-1}a}$$

$$\precsim \int_1^3 \max_{t\in[1,3]} |x(u)-y(u)|e^2\sqrt{1+a^2}e^{i\tan^{-1}a}du$$

$$\precsim 2e^2 d(x,y)$$

Similarly,

$$d(T^n x, T^n y) \precsim e^{2n}\frac{2^n}{n!}d(x,y).$$

Note that

$$e^{2n}\frac{2^n}{n!}\begin{cases}109 & \text{if } n=2\\ 1987 & \text{if } n=4\\ 1.31 & \text{if } n=37\\ 0.53 & \text{if } n=38.\end{cases}$$

Thus for $\lambda = 0.53, \mu = 0, n = 38$, all conditions of Corollary are satisfied and so T has a unique fixed point, which is the unique solution of the integral equation:

$$x(t) = 4 + \int_1^t (x(u)+u^2)e^{u-1}du, \qquad t\in[1,3]$$

or the differential equation:

$$x'(t) = (x+t^2)^{e^{t-1}}, \qquad t\in[1,3], \qquad x(1)=4.$$

Proposition:

1. A G-metric space $(X,G)$is G-complete if and only if (X,d_G) is a complete metric space. Here we start our work with the following theorem

2. Let(X,G)be a G-metric space, then the following are equivalent.

(i)$x_n$is G-convergent to x.

(ii) $G(x_n, x_n, x) \to 0$, as $n\to\infty$.

(iii) $G(x_n, x, x) \to 0$, as $n\to\infty$

(iv) $G(x_m, x_n, x) \to 0$, as $m, n\to\infty$

3. If$(X,G)$is aG-metric space, then the following are equivalent.

(i) The sequence$(x_n)$is G-Cauchy.

(ii) For every $\epsilon > 0$, there exists $N\in N$ such that $G(x_n, x_m, x_m) < \epsilon$,for all $n,m \geq N$.

4. Let $(X,G),(X',G')$ be two G-metric spaces. Then a function $f: X \to X'$ is G-continuous at a point $x \in X$ if and only if it is G sequentially continuous at x; that is, whenever (x_n) is G-convergent to x, $(f(x_n))$is G-convergent to $f(x_n)$.

5. Let (X,G) be a G-metric space, then the function $G(x,y,z)$ is jointly continuous in all three of its variables.

6. Every G-metric space (X,G) will define a metric space (X,d_G) by

$$d_G(x,y) = G(x,y,y) + G(y,x,x), \forall x,y \in X \quad \text{(i)}$$

Note that if $(X,G)$is a symmetric G-metric space, then

$$d_G(x,y) = 2G(x,y,y), \forall x,y \in X \quad \text{(ii)}$$

However, if $(X,G)$is not symmetric, then it holds by the G-metric properties that

$$\frac{3}{2}G(x,y,y) = d_G(x,y) \leq 3G(x,y,y), \forall x,y \in X \quad \text{(iii)}$$

and that in general these inequalities cannot be improved.

4.1 *Conclusion*

In this study the researcher introduced study complex valued metric spaces and established some fixed point results for mappings satisfying a rational inequality. The idea of complex valued metric spaces can be exploited to define complex valued normed spaces and complex valued Hilbert spaces; additionally, it offers numerous research activities in mathematical analysis, our results complement several significant fixed point theorems of G-metric and extendedb-metricspaces in the frame of crisp mappings. We hope that our presented idea herein will be a source of motivation for other researchers to extend and improve these results suitable for their applications.

List of Symbols

Symbols	
max	Maximum
fdp	Form of functional Dynamic pregrcemimity
sup	Supermom

$clRg$	Comman limit in the range of g property
min	Minimum
KM	Kramoid and chakk
Gv	Geovge and veeramani
$W^{1,1}$	Sobolev space
L^1	Lebesgue on the real line
a.e	Almost every where
Dist	Distance
Inf	Infimum
ps_c	Polaissmale condition
L^p	Lebesgue space
ess	Essential
L^2	Hilbert space
$w_o^{p,p}$	Weighlidsobleve space
Re	Real
Im	Imaginary
ext	Extend
$W^{2,2}$	Sobolev space
L^∞	Essoutiallebesgue space
a.a	Almost all
H^2	Hilbert space

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