# Sobolev embeddings of sharp higher order

**Dr.Isam Eldin Ishag Idris**

University of Kordofan, College of Education

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**Abstract:**

This study aims to extend the sharp version of the Sobolev embedding theorem,by using

a natural extension of the $L(2-\epsilon,1-\epsilon)$ spaces with a new form of the Pólya-Szegö symmetrization principle, it follows the descriptive method, and the study found that

$Y_{2+\epsilon}(\infty,1+\epsilon)$ is not larger than any $r.s.$ space $X(\Omega)$ such that $W_0^{(1+\epsilon,Y)}(\Omega)\subset X(\Omega)$,this result extends, complements, simplifies and sharpens all the recent results.

**Keywords**: Embedding, Sobolev rearrangement

**المستخلص:**

هدفت هذه الدراسة إلى العمل على تمديد نسخة من نظرية تضمين سوبوليف الحدية بإستخدام الإمتداد الطبيعى للفضاءات $L(2-\epsilon,1-\epsilon)$ مع إطارجديد لقواعد التناظر التى صاغها بولا-سزيقو واتبعت الدراسة المنهج الوصفى وتوصلت إلى النتيجة إن فضاء $Y_{2+\epsilon}(\infty,1+\epsilon)$ ليس أكبرمن أى فضاء $r.s.$ بحيث أن $W_0^{(1+\epsilon,Y)}(\Omega)\subset X(\Omega)$ وتعمل هذه النتيجة على تمديد و تكملة وتبسيط حدية كل النتائج اللاحقة.

**كلمات مفتاحية: تضمين،سوبوليف،رتيبى ثابت**

## 1.Introduction

We show that $W_0^{(1+\epsilon,2-\epsilon)}(\Omega)$ denote the usual Sobolev spaces of functions $\varphi$ defined on an open domain $\Omega\subset\mathbb{R}^{(2+\epsilon)}$, such that $\varphi$ and all its distributional derivatives $D^\alpha\varphi,|\alpha|\leq 1+\epsilon$, are zero at $\partial\Omega$, and, moreover, such that $|D^\alpha\varphi|\in L^P(\Omega),|\alpha|=1+\epsilon$. The classical Sobolev embedding theorem asserts that if $\Omega$ an open domain in $\mathbb{R}^{(2+\epsilon)}$ then,

$$W_0^{(1+\epsilon,2-\epsilon),}(\Omega) \subset L^{(1-\epsilon)}(\Omega), \epsilon^3 - \epsilon^2 - \epsilon - 2 = 0 ,1 > \epsilon > 0 . \quad (1.1)$$

The norms of the embedding blow up as $2 - \epsilon \to \frac{2+\epsilon}{1+\epsilon}$. In fact, if $\epsilon = 0$, we formally have $\epsilon = \infty$ and (1.1) is false. Thus, in the limiting case , it is necessary to go outside the $L^{(1-\epsilon)}$ scale to find the correct target spaces.Indeed, it was shown by ([17] Trudinger, (1967) .

for $\epsilon = 0$, and ([19] Strichartz ,1971/72). for $\epsilon > 0$ (cf. also ([15] Cwikel and Pustylnik ,1998), if $|\Omega| < \infty$, we have

$$W_0^{\left(1+\epsilon,\frac{2+\epsilon}{1+\epsilon}\right)}(\Omega) \subset L_{\Phi_{\left(\frac{2+\epsilon}{1+\epsilon}\right)}}(\Omega), \quad (1.2)$$

where $\Phi_{\left(\frac{2+\epsilon}{1+\epsilon}\right)}(x)$ is a Young's function such that $\Phi_{\left(\frac{2+\epsilon}{1+\epsilon}\right)}(x) \approx e^{x^{\left(\frac{2+\epsilon}{2\epsilon-1}\right)}}$ for large $x$. It is also known that neither (1.1) nor (1.2) are sharp. The sharp form of (1.1) is provided by the O'Neil-Stein version of the Sobolev embedding theorem (cf. [18] O'Neil,1963).

which requires the use of the $L(2-\epsilon, 1-\epsilon)$ spaces :

$$W_0^{(1+\epsilon,2-\epsilon)}(\Omega) \subset L(2-\epsilon, 1-\epsilon)(\Omega), \epsilon^3 - \epsilon^2 - \epsilon - 2 = 0 ,1 > \epsilon > 0 . \quad (1.3)$$

Again (1.3) fails when $\epsilon = 0$. Motivated in part by (1.3), ([14] Hansson, (1979)

and ([10] Brézis-Wainger,1980) improved on (1.2) and obtained in the limiting case $\epsilon = 0$,

$$W_0^{\left(1+\epsilon,\frac{2+\epsilon}{1+\epsilon}\right)}(\Omega) \subset \mathrm{H}_{\left(\frac{2+\epsilon}{1+\epsilon}\right)}(\Omega), \quad (1.4)$$

where, for $\epsilon < 0$, $\mathrm{H}_{(1-\epsilon)}(\Omega)$ is the Lorentz space defined by

$$\mathrm{H}_{(1-\epsilon)}(\Omega)=\left\{\varphi:\|\varphi\|_{\mathrm{H}_{(1-\epsilon)}(\Omega)}\right.$$

$$=\left\{\int_0^{|\Omega|}\left(\frac{\varphi^{**}(1+\epsilon)}{1+\ln\frac{|\Omega|}{1+\epsilon}}\right)^{(1-\epsilon)}\frac{d(1+\epsilon)}{1+\epsilon}\right\}^{\left(\frac{1}{1-\epsilon}\right)}<\infty\Bigg\}.$$

The result is optimal amoung all $r.i.$ spaces; this was proved in ([14] Hansson,1979).

for Riesz potentials and in ([15] Cwikel and Pustylnik ,1998) for Sobolev spaces themselves.

In particular,in ([15] Cwikel and Pustylnik ,1998) it is shown that, for any $r.i.$ spaces $X(\Omega)$,

$$W_0^{\left(1+\epsilon,\frac{2+\epsilon}{1+\epsilon}\right)}(\Omega)\subset X(\Omega)\Rightarrow \mathrm{H}_{\left(\frac{2+\epsilon}{1+\epsilon}\right)}(\Omega)\subset X(\Omega).\quad (1.5)$$

It is customary to treat (1.3) and (1.4) (or (1.2)) as separate problems with their corresponding separate proofs. We shall show extending the methods developed in ([11] Bastero, Milman and Ruiz) for the case $\epsilon=0$, a unified method to prove the Sobolev embedding theorem and the corresponding sharp borderline cases. In fact, for the borderline cases, we actually improve on the classical results since our target spaces are rearrangement invariant sets which are strictly contained in the optimal spaces described above.

The two main ingredients of our method are : (a) the use of a very natural extension of the $L(2-\epsilon,1-\epsilon)$ spaces recently proposed in ([11] Bastero, Milman and Ruiz), and (b) the use of a newversion of the Pólya-Szegö symmetrization principle that is valid for higher order derivatives. The new method can be easily generalized to give sharp results in the context of Sobolev spaces based on general rearrangement invariant spaces. To better explain the ingredients of the proof we start by recalling the $L(\infty, 1-\epsilon)$ spaces recently introduced in ([11] Bastero, Milman and Ruiz) in connection with the case $\epsilon = 0$ of (1.3). We observe that if we formally let $2-\epsilon \to \frac{2+\epsilon}{1+\epsilon}$ in (1.3), and disregard the blow up constants, one is naturally led to consider the space $L\left(\infty, \frac{2+\epsilon}{1+\epsilon}\right)$. However, it is well known, and easy to see,that with the usual definition $L(\infty, 1-\epsilon) = \{0\}$. On the other hand, imitating ([4] Bennett-DeVore-Sharpley,1981), we modify the definition of the $L(2-\epsilon, 1-\epsilon)$ spaces as follows. Let $1 > \epsilon \geq \infty, 1 < \epsilon \leq \infty$, ℃, let $\Omega$ be a domain in $R^{(2+\epsilon)}$, and let $M(\Omega)$ be the set of measurable functions on $\Omega$.

We let $L(2-\epsilon, 1-\epsilon)(\Omega) = \left\{\varphi \in M(\Omega): \|\varphi\|_{L(2-\epsilon,1-\epsilon)} \leq \infty\right\}$, with

$$\|\varphi\|_{L(2-\epsilon,1-\epsilon)} = \left\{\int_0^\infty \left[\left(\varphi^{**}(1-\epsilon) - \varphi^*(1-\epsilon)\right)(1-\epsilon)^{\left(\frac{1}{2-\epsilon}\right)}\right]^{(1-\epsilon)} \frac{d(1-\epsilon)}{1-\epsilon}\right\}^{\left(\frac{1}{1-\epsilon}\right)},$$

$1 > \epsilon \geq \infty, 0 > \epsilon > \infty$ we use the usual modifications when $\epsilon = \infty$.

These spaces make sense (and are not trivial) for $1 > \epsilon \geq \infty, 1 < \epsilon \leq \infty$. Moreover, the new spaces actually coincide with the classical $L(2-\epsilon, 1-\epsilon)$ spaces whenever

$1 > \epsilon > \infty, 1 < \epsilon \leq \infty$ In fact for functions in the classical $L(2-\epsilon, 1-\epsilon)$ spaces we have

$$\left\{\int_0^\infty \left[\left(\varphi^{**}(1-\epsilon)-\varphi^{*}(1-\epsilon)\right)(1-\epsilon)^{\left(\frac{1}{2-\epsilon}\right)}\right]^{(1-\epsilon)} \frac{d(1-\epsilon)}{1-\epsilon}\right\}^{\left(\frac{1}{1-\epsilon}\right)}$$

$$\approx \left\{\int_0^\infty \left[\left(\varphi^{**}(1-\epsilon)\right)(1 - \epsilon)^{\left(\frac{1}{2-\epsilon}\right)}\right]^{1-\epsilon} \frac{d(1-\epsilon)}{1-\epsilon}\right\}^{\left(\frac{1}{1-\epsilon}\right)}, 1>\epsilon>\infty, 1<\epsilon\leq\infty.$$

The second idea underlying our method is a suitable extension of the Pólya-Szegö symmetrization principle for higher order derivatives. It is easy to see that the standard form of this principle, comparing the first derivatives of a function $\varphi(x)$ and its non increasing rearrangement $\varphi^{*}(1-\epsilon)$, can not be generalized, even to second derivatives, because there are infinitely many smooth functions $\varphi(x)$ such that $\frac{d\varphi^{*}}{d(1-\epsilon)}$ is not differentiable, even in the sense of distributions. For example,the function $\varphi(x)=1+\sin x, 0<x<\frac{3\pi}{2}$, has rearrangement $\varphi^{*}(1-\epsilon)=\left(1+\cos\frac{1-\epsilon}{2}\right)\chi_{(0,\pi)}(1-\epsilon)+(1+\sin(1-\epsilon))\chi_{\left[\pi,\frac{3\pi}{2}\right)}(1-\epsilon)$, thus $\frac{d\varphi^{*}}{d(1-\epsilon)}$ has a ‘‘jump’’ at the point $1-\epsilon=\pi$ (a more detailed discussion devoted to this topic can be found in ([1] Cianchi, 2000).Nevertheless, we shall show that a suitable modification of Pólya-Szegö holds for higher order derivatives.To state our result, we recall an inequality for radial spherically decreasing rearrangements from ([11] Bastero, Milman and Ruiz)and ([3] Alvino, Trombetti and Lions,1989). Suppose that $\varphi$ is a smooth function such that $\varphi$ and $|\nabla\varphi|$ vanish at inifinity and let $\varphi^{\circ}$ denote its radial spherically decreasing symmetric rearrangement. Then, there exists a universal constant $\gamma_{(2+\epsilon)}$ such that

$$(1-\epsilon)^{\left(\frac{-1}{2+\epsilon}\right)}\left(\varphi^{**}(1-\epsilon)-\varphi^{*}(1-\epsilon)\right) \leq \gamma_{(2+\epsilon)}\left|\nabla\varphi^{\circ}\right|^{**}(1-\epsilon). \quad (1.6)$$

This leads to

$$\left\|(1-\epsilon)^{\left(\frac{-1}{2+\epsilon}\right)}\left(\varphi^{**}(1-\epsilon)-\varphi^{*}(1-\epsilon)\right)\right\|_{Y} \lesssim \|\nabla\varphi\|_{Y}, \forall\ \varphi \in C_0^{\infty}\left(\mathbb{R}^{(2+\epsilon)}\right) \quad (1.7)$$

for any $r.i.$ space $Y$ satisfying the $(P)$ condition (cf.Definition 2.1below). We will show (cf Theorem 3.4 below) that, under mild conditions on a $r.i.$ norm

$$Y(\Omega), 0 \leq \epsilon \leq 1,$$

$$\left\|(1-\epsilon)^{\left(\frac{-(1+\epsilon)}{2+\epsilon}\right)}\left(\varphi^{**}(1-\epsilon)-\varphi^{*}(1-\epsilon)\right)\right\|_{Y(\Omega)} \lesssim \left\|\nabla^{(1+\epsilon)}\varphi\right\|_{Y(\Omega)}, \forall\ \varphi \in C_0^{\infty}\left(\mathbb{R}^{(2+\epsilon)}\right). \quad (1.8)$$

We prove that conditions of this type are optimal by means of reformulating a necessary condition for Sobolev emeddings derived in ([6] Edmunds,Kerman and Pick,2000).

(cf.Theorem (3.6). More precisely we show that if $X(\Omega)$ is a $r.i.$ space such that

$\|\varphi\|_{X(\Omega)} \lesssim \left\|\nabla^{(1+\epsilon)}\varphi\right\|_{Y(\Omega)}$ and $|\Omega| < \infty$, then

$$\|\varphi^{*}\|_{X(\Omega)} \leq c(\Omega)\left\|(1-\epsilon)^{\left(\frac{-(1+\epsilon)}{2+\epsilon}\right)}\left(\varphi^{**}(1-\epsilon)-\varphi^{*}(1-\epsilon)\right)\right\|_{Y}. \quad (1.9)$$

We are now ready to give our proof of the following sharp form of the Sobolev embedding theorem.

Theorem 1.1 Let $\Omega$ be an open domain in $\mathbb{R}^{(2+\epsilon)}$, let $\epsilon \geq -2, \epsilon \geq 0, 1 > \epsilon > 0$, with the convention that $\epsilon = \infty$ if $\frac{1}{1-\epsilon} = 0$ (i.e. $\epsilon = \infty$ when $= \frac{1+\epsilon}{2+\epsilon}$) .Then

$$W_0^{\left(1+\epsilon,\frac{2+\epsilon}{1+\epsilon}\right)}(\Omega) \subset L(1-\epsilon, 2-\epsilon)(\Omega), \quad (1.10)$$

with

$$\|\varphi\|_{L(1-\epsilon,2-\epsilon)} \leq c\left\|\nabla^{(1+\epsilon)}\varphi\right\|_{L(1-\epsilon)}, \forall \varphi \in C_0^{\infty}(\Omega). \quad (1.11)$$

Moreover, the result is optimal : if $|\Omega| < \infty$, then for any $r.i.$ space $X(\Omega), 1 > \epsilon > 0, W_0^{(1+\epsilon,2-\epsilon)}(\Omega) \subset X(\Omega) \Rightarrow L\left(\frac{4-\epsilon^2}{-\epsilon(\epsilon+2)}, 2-\epsilon\right) \subset X(\Omega)$.

Proof. As a consequence of Example 4.1it follows that (1.8) holds with $Y = L^{(2-\epsilon)}, 1 > \epsilon > 0$. Therefore (1.10) and (1.11) follow from the following trivial computation with indices,

$$L\left(\frac{4-\epsilon^2}{-\epsilon(\epsilon+2)}, 2-\epsilon\right) = \left\{\varphi: \left(\varphi^{**}(1-\epsilon) - \varphi^{*}(1-\epsilon)\right)(1-\epsilon)^{\left(\frac{-(1+\epsilon)}{2+\epsilon}\right)} \in L^{(2-\epsilon)}\right\}, 1 > \epsilon > 0,$$

with $\|\varphi\|_{L\left(\frac{4-\epsilon^2}{-\epsilon(\epsilon+2)}, 2-\epsilon\right)} = \left\|\left(\varphi^{**}(1-\epsilon) - \varphi^{*}(1-\epsilon)\right)(1-\epsilon)^{\left(\frac{-(1+\epsilon)}{2+\epsilon}\right)}\right\|_{L(2-\epsilon)}$.

That this condition is optimal now follows directly from the previous calculation and (1.9).

To extend Theorem 1.1 to more general $r.i.$ spaces, it is therefore natural to turn the left hand side of (1.8) into a definition. Given $r.i.$ space $Y(\Omega)$, let

$$Y_{(2+\epsilon)}\big(\infty,(1+\epsilon)\big)(\Omega) = \left\{\varphi : (1-\epsilon)^{\left(\frac{-(1+\epsilon)}{2+\epsilon}\right)}\big(\varphi^{**}(1-\epsilon)-\varphi^{*}(1-\epsilon)\big) \in Y(\Omega)\right\},$$

$$\|\varphi\|_{Y_{(2+\epsilon)}(\infty,(1+\epsilon))} = \left\|(1-\epsilon)^{\left(\frac{-(1+\epsilon)}{2+\epsilon}\right)}\big(\varphi^{**}(1-\epsilon) - \varphi^{*}(1-\epsilon)\big)\right\|_{Y}. \quad (1.12)$$

The preceding discussion leads to the following generalization of Theorem 1.1.

Theorem 1.2. Let $\Omega$ be an open domain in $\mathbb{R}^{(2+\epsilon)}$, suppose that $\epsilon < \frac{1}{2}$. Let $Y(\Omega)$ be a $r.i.$ space satisfying the conditions $Q(\epsilon)$ (cf. Definition 2.2 below) and (2.1 below).Then

$$W_0^{(1+\epsilon),Y}(\Omega) \subset Y_{(2+\epsilon)}\big(\infty,(1+\epsilon)\big), \quad (1.13)$$

and $\|\varphi\|_{Y_{(2+\epsilon)}(\infty,(1+\epsilon))} \lesssim \left\|\nabla^{(1+\epsilon)}\varphi\right\|_{Y(\Omega)}, \forall\ \varphi \in C_0^{\infty}(\Omega)$.

Moreover, if $|\Omega| < \infty$, and $X(\Omega)$ is a $r.i.$ space then $W_0^{(1+\epsilon,Y)}(\Omega) \subset X(\Omega) \Rightarrow Y_{(2+\epsilon)}\big(\infty,(1+\epsilon)\big) \subset X(\Omega)$, and $\|\varphi^{**}\|_X \lesssim \|\varphi\|_{Y_{(2+\epsilon)}(\infty,(1+\epsilon))}$.

Proof. If $Y$ is a $r.i.$ space satisfying the assumptions of the theorem then, by Lemma 2.3 and Theorem 3.4 below, (1.8) holds. This is all we need to obtain (1.13). The fact that the

$Y_{(2+\epsilon)}(\infty,(1+\epsilon))$ spaces are optimal follows once again from (1.9).

The last theorem obtains a particularly simple from if the space $Y_{(2+\epsilon)}(\infty,(1+\epsilon))$ itself is a $r.i.$ space. For example, if $Y$ satisfies the $Q(1+\epsilon)$ - condition (see Definition 2.2 below), it follows by Lemma 2.6 that

$$\|\varphi\|_{Y_{(2+\epsilon)}(\infty,(1+\epsilon))} \approx \left\|(1-\epsilon)^{\left(\frac{-(1+\epsilon)}{2+\epsilon}\right)}\varphi^{**}(1-\epsilon)\right\|_Y . \quad (1.14)$$

Coroally1.3. Let $\Omega$ be an open domain in $\mathbb{R}^{(2+\epsilon)}$, let $Y(\Omega)$ be a $r.i.$ space satisfying the $Q(1+\epsilon)$ - condition .Then $Y_{(2+\epsilon)}(\infty, 1+\epsilon)$ is a $r.i.$ space with norm provided by (1.14) and moreover, $\left\|(1-\epsilon)^{\left(\frac{-(1+\epsilon)}{2+\epsilon}\right)}\varphi^{*}(1-\epsilon)\right\|_Y \lesssim \|\varphi\|_{Y_{(2+\epsilon)}(\infty,(1+\epsilon))} \lesssim \|\varphi\|_{W_0^{(1+\epsilon),Y}(\Omega)}$.

$Y_{(2+\epsilon)}(\infty,(1+\epsilon))$ is the optimal target space for the Sobolev embedding $W_0^{(1+\epsilon),Y}(\Omega) \subset X(\Omega)$ among all $r.i.$ spaces. The quasi-normed space defined by the quasi-norm $\left\|(1-\epsilon)^{\left(\frac{-(1+\epsilon)}{2+\epsilon}\right)}\varphi^{*}(1-\epsilon)\right\|_Y$ is the optimal target space among the class of quasi-normed $r.i.$ spaces ([16] Milman and Pustylnik, 2004).

**Remark 1.4.** The optimal target spaces for embeddings of generalized Sobolev spaces are described in ([6] Edmunds,Kerman and Pick,2000).The description obtained in ([6] Edmunds,Kerman and Pick,2000) is indirect and does not imply the previous Corollary. A version of our Corollary 1.3 ( with stronger assumptions on Y and without a study of optimal conditions) was claimed much earlier in ([24] Klimov, (1970).Unfortunately, the proof indicated there is based on two incorrect arguments (a higher order Pólya-Szegö principle as a mere iteration of first order results, and a theorem on interpolation of $r.i.$ spaces that was later shown to be false).

**2. Preliminaries**

Let $\Omega \subset \mathbb{R}^{(2+\epsilon)}$ be a domain, and let $Y = Y(\Omega)$ be rearrangement invariant space. Let $1+\epsilon \in \mathbb{N}$ and define the Sobolev spaces

$W_0^{(1+\epsilon),Y}(\Omega) = \{\varphi: |D^\alpha \varphi| \in Y, D^\alpha \varphi$ vanishes on $\partial\Omega, |\alpha| \leq 1+\epsilon\}$,

where $|D^\alpha \varphi|$ is the length of the vector whose components are all the derivatives of $\varphi$ of order $|\alpha|$. $W_0^{(1+\epsilon),Y}(\Omega)$ is provided with the seminorm

$\|\varphi\|_{W_0^{(1+\epsilon),Y}(\Omega).} = \sum_{|\alpha|=1+\epsilon} \|D^\alpha \varphi\|_Y.$ Let

$\nabla\varphi = \left(\frac{\partial\varphi}{\partial x_1}, \dots, \frac{\partial\varphi}{\partial x_{(2+\epsilon)}}\right), \nabla\varphi = \sum_{i=1}^{(2+\epsilon)} \frac{\partial^2\varphi}{\partial x_i^2}$ and

$$\nabla^{(1+\epsilon)}\varphi = \begin{cases} \Delta^{\left(\frac{1+\epsilon}{2}\right)}\varphi & \text{for even } (1+\epsilon), \\ \nabla\left(\Delta^{\left(\frac{\epsilon}{2}\right)}\right)\varphi & \text{for odd } (1+\epsilon). \end{cases}$$

We shall usually formulate conditions on $r.i.$ spaces $Y$ in terms of the Hardy operators $P\varphi(1-\epsilon) = \frac{1}{1-\epsilon}\int_0^{(1-\epsilon)} \varphi(1+\epsilon)\, d(1+\epsilon)$; $\quad Q\varphi(1-\epsilon) = \int_{(1-\epsilon)}^{\infty} \varphi(1+\epsilon)\frac{d(1+\epsilon)}{1+\epsilon}$.

Recall that a $r.i.$ space has a representation as a function space on $Y^{\hat{}}(0,|\Omega|)$ such that $\|\varphi\|_{Y(\Omega)} = \|\varphi^*\|_{Y^{\hat{}}(0,|\Omega|)}$. Moreover, since the measure space will be always clear from the context it is convenient to "drop the hat" and use the same letter $Y$ to indicate the different versions of the space $Y$ that we use. We shall also set our functions equal to zero for $x \notin \Omega$. Let

$(\sigma_{(1+\epsilon)}\varphi)(1-\epsilon) = \varphi(\frac{1-\epsilon}{1+\epsilon}), 1-\epsilon, \epsilon > -1,$ and define the dilation function $d_Y(1+\epsilon)$ by $d_Y(1+\epsilon) = \|(\sigma_{(1+\epsilon)}\varphi)(1-\epsilon)\|_Y = \|(\sigma_{(1+\epsilon)}\varphi^*)(1-\epsilon)\|_Y$
(since $(\sigma_{(1+\epsilon)}\varphi)^* = (\sigma_{(1+\epsilon)}\varphi^*)$) .

Definition 2.1.We shall say that $Y$ satisfies the $(P)$-condition if $P: Y(0,\infty) \to Y(0,\infty)$ defines a bounded operator (equivalently $Y$ satisfies the $(P)$-condition if and only if the upper Boyd index of $Y$ is less than 1) . In particular, if $Y$ satisfies the $(P)$-condition then $\|\varphi^{**}\|_Y$ is an equivalent norm on $Y$,

$$\|\varphi\|_Y \approx \|\varphi^{**}\|_Y. \quad (2.1)$$

Likewise we shall say that $Y$ satisfies the $(Q)$ -condition if $Q: Y(0,\infty) \to Y(0,\infty)$ defines a bounded operator (equivalently $Y$ satisfies the $(Q)$-condition if and only if the lower Boyd index of $Y$ is greater than 0).

In what follows we also need to consider weighted norm inequalities for $Q$ with power weights. This leads to conditions on our spaces.

Definition 2.2. Let $\epsilon > -1$, and let $Y$ be a $r.i.$ space. We shall say that $Y$ satisfies the $Q(1+\epsilon)$-condition if

$$Q((1+\epsilon), Y) = \int_1^\infty (1+\epsilon)^{\left(\frac{1+\epsilon}{2+\epsilon}\right)} d_Y\left(\frac{1}{1+\epsilon}\right)\frac{d(1+\epsilon)}{1+\epsilon} < \infty. \quad (2.2)$$

Note that if $Y$ satisfies the $Q((1+\epsilon))$ -condition for some $(1+\epsilon)$ then it satisfies a $Q(b)$-condition for every $b < 1+\epsilon$, including the $Q(0) = (Q)$-condition.The following known result (cf. ([8] Stein , 1970, [20] Maz'ya ,1985)) will be useful in what follows.

Lemma 2.3. Let $\Omega$ be an open domain in $\mathbb{R}^{(2+\epsilon)}$, and let $Y(\Omega)$ be a $r.i.$ space satisfying the$(P)$ and $(Q)$-conditions.Then

$$\|\Delta\varphi\|_{Y(\Omega)} \approx \sum_{|\alpha|=2} \|D^{\alpha}(\varphi)\|_{Y(\Omega)}, \forall\, \varphi \in C_0^{\infty}(\Omega), \quad (2.3)$$

holds with constants of equivalence independent of $\varphi$.

Proof. We only need to remark that the arguments in ([8] Stein , 1970,[20] Maz'ya ,1985) for $L^{(2-\epsilon)}$ extend to $r.i.$ spaces satisfying the$(P)$ and $(Q)$-conditions by a well known result due to Boyd (cf.[5] Bennett, DeVore and Sharpley,1988).

Example 2.4. In particular if $\Omega$ is an open domain in $\mathbb{R}^{(2+\epsilon)}$, $1 > \epsilon > \infty$, then $\|\Delta\varphi\|_{L^{(2-\epsilon)}(\Omega)} \approx \sum_{|\alpha|=2}\|D^{\alpha}(\varphi)\|_{L^{(2-\epsilon)}(\Omega)}, \forall\, \varphi \in C_0^{\infty}(\Omega)$.

Lemma 2.5. If $Y$ is a $r.i.$ space satisfying the $Q(1+\epsilon)$ -condition, then $Q$ is bounded on the space $Y$ provided with weight $(1-\epsilon)^{\left(\frac{-(1+\epsilon)}{2+\epsilon}\right)}$. More precisely,

$$\left\|(1-\epsilon)^{\left(\frac{-(1+\epsilon)}{2+\epsilon}\right)} Q\varphi(1-\epsilon)\right\|_{Y} \leq Q\big((1+\epsilon), Y\big)\left\|(1-\epsilon)^{\left(\frac{-(1+\epsilon)}{2+\epsilon}\right)} \varphi(1-\epsilon)\right\|_{Y}. \quad (2.4)$$

Proof. Let $\alpha = \frac{1+\epsilon}{2+\epsilon}$. We have

$$(1-\epsilon)^{-\alpha}|Q\varphi(1-\epsilon)| = \int_{(1-\epsilon)}^{\infty} \varphi(1+\epsilon)\,(1-\epsilon)^{-\alpha}\frac{d(1+\epsilon)}{1+\epsilon}$$

$$= \int_{1}^{\infty} \varphi(1-\epsilon^2)\,(1-\epsilon)^{-\alpha}\frac{d(1+\epsilon)}{1+\epsilon}$$

$$= \int_{1}^{\infty} \varphi(1-\epsilon^2)\,(1-\epsilon^2)^{-\alpha}(1+\epsilon)^{\alpha-1}{}_{(1-\epsilon)}d(1+\epsilon).$$

Applying Minkowski's inequality we obtain

$$\|(1-\epsilon)^{-\alpha}Q\varphi(1-\epsilon)\|_Y \le \int_{1}^{\infty} \|\varphi(1-\epsilon^2)(1-\epsilon^2)^{-\alpha}\|_Y\,(1+\epsilon)^{\alpha-1}{}_{(1-\epsilon)}d(1+\epsilon)$$

$$\le \int_{1}^{\infty} d_Y\left(\frac{1}{1+\epsilon}\right)(1+\epsilon)^{\alpha-1}{}_{(1-\epsilon)}d(1+\epsilon)\|(1-\epsilon)^{-\alpha}\varphi\|_Y.$$

Lemma 2.6. Let $Y$ be a $r.i.$ space satisfying the $Q(1+\epsilon)$ -condition, for some $\epsilon > -1$. Then, for all measurable functions $\varphi$ with $\varphi^{**}(\infty) = 0$,

$$\left\|(1-\epsilon)^{\left(\frac{-(1+\epsilon)}{2+\epsilon}\right)}(\varphi^{**}(1-\epsilon) - \varphi^{*}(1-\epsilon))\right\|_Y \le \left\|(1-\epsilon)^{\left(\frac{-(1+\epsilon)}{2+\epsilon}\right)}\varphi^{**}(1-\epsilon)\right\|_Y \le Q(1+\epsilon, Y)\left\|(1-\epsilon)^{\left(\frac{-(1+\epsilon)}{2+\epsilon}\right)}(\varphi^{**}(1-\epsilon) - \varphi^{*}(1-\epsilon))\right\|_Y. \quad (2.5)$$

Proof. The first inequality is trivial. To prove the second inequality note that

$\frac{d}{d(1-\epsilon)}\varphi^{**}(1-\epsilon)=\frac{\varphi^{*}(1-\epsilon)-\varphi^{**}(1-\epsilon)}{1-\epsilon}$. Therefore, by the fundamental theorem of calculus, we have

$$\varphi^{**}(1-\epsilon)=\int_{(1-\epsilon)}^{\infty}\left(\varphi^{**}(1+\epsilon)-\varphi^{*}(1+\epsilon)\right)\frac{d(1+\epsilon)}{1+\epsilon}$$
$$=Q(\varphi^{**}-\varphi^{*})(1-\epsilon).\,(2.6)$$

The desired result follows from Lemma 2.5.

It is useful to remark here that, for the operator $P$, the corresponding weighted norm inequalities for power weights are automatically true.

Lemma 2.7. Let $Y$ be a $r.i.$ space. Then, for any $\alpha>0$

$$\|(1-\epsilon)^{-\alpha}P(\varphi)(1-\epsilon)\|_Y\leq\frac{1}{\alpha}\|(1-\epsilon)^{-\alpha}\varphi(1-\epsilon)\|_Y.$$

Proof. Computing $d_Y(1+\epsilon)$ for $Y=L^1$, and $Y=L^\infty$,we find, by interpolation, that for any $r.i.$ space $Y\,d_Y(1+\epsilon)=\max\{1,1+\epsilon\}$ (cf.[5] Bennett, DeVore and Sharpley,1988 for the details). Since $P$ is a positive operator we may suppose that $\varphi\geq 0$. Now,

$(1-\epsilon)^{-\alpha}P\varphi(1-\epsilon)=\frac{1}{1-\epsilon}\int_0^{(1-\epsilon)}\varphi(1+\epsilon)\,(1-\epsilon)^{-\alpha}d(1+\epsilon)=\int_0^1\varphi(1-\epsilon^2)\left((1+\epsilon)(1-\epsilon)\right)^{-\alpha}(1+\epsilon)^{\alpha}d(1+\epsilon)$.Thus,

$$\|(1-\epsilon)^{-\alpha}P\varphi(1-\epsilon)\|_Y$$
$$\leq\int_0^1 d_Y\left(\frac{1}{1+\epsilon}\right)(1+\epsilon)^{\alpha}d(1+\epsilon)\|(1-\epsilon)^{-\alpha}\varphi(1-\epsilon)\|_Y$$
$$\leq\int_0^1(1+\epsilon)^{\alpha-1}d(1+\epsilon)\,\|(1-\epsilon)^{-\alpha}\varphi(1-\epsilon)\|_Y,$$

as desired.

**3.Symmetrization inequalities for higher order Sobolev spaces**

The main tool for our analysis is the following result from ([11] Bastero, Milman and Ruiz).

Lemma 3.1. Suppose that $Y$ is a $r.i.$ space satisfying a $(P)$-condition. Then for all smooth functions $\varphi$ such that $\varphi^{**}(\infty) = 0$,

$$\left\|(1-\epsilon)^{\left(\frac{-1}{2+\epsilon}\right)}\left(\varphi^{**}(1-\epsilon)-\varphi^{*}(1-\epsilon)\right)\right\|_{Y} \lesssim \|\nabla\varphi\|_{Y}. (3.1)$$

Proof. The proof we give is the same as the one given in ([11] Bastero, Milman and Ruiz) for $Y = L^{(2-\epsilon)}$. However, it is important for our development to provide the complete details here. Recall that from Lemma 1 in ([11] Bastero, Milman and Ruiz) we have the pointwise inequality

$$(1-\epsilon)^{\left(\frac{-1}{2+\epsilon}\right)}\left(\varphi^{**}(1-\epsilon)-\varphi^{*}(1-\epsilon)\right) \lesssim \left|\nabla\varphi^{\circ}\right|^{**}(1-\epsilon), (3.2)$$

where $\varphi^{\circ}(x) = \varphi^{*}\left(c_{(2+\epsilon)}|x|^{(2+\epsilon)}\right)$, denotes the radial spherically symmetrically decreasing rearrangement of $\varphi$, $c_{(2+\epsilon)}$ =measure of the unit ball.Recall also that the Pólya-Szegö symmetrization principle holds for r.i. spaces (cf.[12] Fournier, 1987, [23] Klimov,1969)

$$\left\|\nabla\varphi^{\circ}\right\|_{Y} \lesssim \|\nabla\varphi\|_{Y}. (3.3)$$

Applying the $Y$ norm to (3.2), and using successively the $(P)$-condition and (3.3), we obtain the desired result.

Our main result here is the higher order version of Lemma 3.1.The first step of the induction process that leads to higher order estimates is provided by the next result.

Theorem 3.2. Let $\Omega$ be a domain in $\mathbb{R}^{(2+\epsilon)}$, and let $Y(\Omega)$ be a $r.i.$ space satisfying the $(P)$ and $Q(1)$ conditions. Then for all smooth functions $\varphi$ such that

$$\varphi^{**}(\infty) = |\nabla\varphi|^{**}(\infty) = 0,$$

$$\left\|(1-\epsilon)^{\left(\frac{-2}{2+\epsilon}\right)}\left(\varphi^{**}(1-\epsilon)-\varphi^{*}(1-\epsilon)\right)\right\|_{Y} \lesssim \|\nabla\varphi\|_{Y}. (3.4)$$

Proof. We shall prove below the elementary estimate

$$|\nabla||\nabla\varphi| \leq |D^2\varphi|. \ (3.5)$$

Applying Lemma 3.1 to $|\nabla\varphi|$ and combining with (3.5), we obtain

$$\left\|(1-\epsilon)^{\left(\frac{-1}{2+\epsilon}\right)}\left(|\nabla\varphi|^{**}(1-\epsilon)-|\nabla\varphi|^{*}(1-\epsilon)\right)\right\|_{Y} \lesssim$$

$\||\nabla\varphi|\nabla\varphi\|_{Y} \lesssim \|D^2\varphi\|_{Y}$.

Therefore combining with (2.3) we get

$$\left\|(1-\epsilon)^{\left(\frac{-1}{2+\epsilon}\right)}\left(|\nabla\varphi|^{**}(1-\epsilon)-|\nabla\varphi|^{*}(1-\epsilon)\right)\right\|_{Y} \lesssim \|\Delta\varphi\|_{Y}.$$

Combining the previous inequality with Lemma 2.6 we obtain

$$\left\|(1-\epsilon)^{\left(\frac{-1}{2+\epsilon}\right)}|\nabla\varphi|^{**}(1-\epsilon)\right\|_{Y} \lesssim \|\Delta\varphi\|_{Y}.$$

By the generalized Pólya-Szegö symmetrization principle (3.3) (applied to the $r.i.$ norm defined by $\|\psi\|_{B} = \left\|(1-\epsilon)^{\left(\frac{-1}{2+\epsilon}\right)}\psi^{**}(1-\epsilon)\right\|_{Y}$) we see that

$$\left\|(1-\epsilon)^{\left(\frac{-1}{2+\epsilon}\right)}|\nabla\varphi^{\circ}|^{**}(1-\epsilon)\right\|_{Y} \lesssim \left\|(1-\epsilon)^{\left(\frac{-1}{2+\epsilon}\right)}|\nabla\varphi|^{**}(1-\epsilon)\right\|_{Y} \lesssim \|\Delta\varphi\|_{Y}. (3.6)$$

Combining the last inequality with the pointwise inequality (3.2) we find

$$\left\|(1-\epsilon)^{\left(\frac{-2}{2+\epsilon}\right)}\left(\varphi^{**}(1-\epsilon)-\varphi^{*}(1-\epsilon)\right)\right\|_{Y} \lesssim \left\|(1-\epsilon)^{\left(\frac{-1}{2+\epsilon}\right)}|\nabla\varphi^{\circ}|^{**}(1-\epsilon)\right\|_{Y}.$$

Inserting the last estimate in (3.6) gives us desired estimate

$$\left\|(1-\epsilon)^{\left(\frac{-2}{2+\epsilon}\right)}\left(\varphi^{**}(1-\epsilon)-\varphi^{*}(1-\epsilon)\right)\right\|_{Y} \lesssim \|\Delta\varphi\|_{Y}.$$

It remains to prove (3.5). Using subindexes to indicate partial derivatives, we have

$$|\nabla\psi|_{j} = \frac{1}{|\nabla\psi|}\sum_{i=1}^{(2+\epsilon)} \psi_i\psi_{ij}\, |\nabla|\nabla\psi||$$

$$= \frac{1}{|\nabla\psi|}\left(\sum_{j=1}^{(2+\epsilon)}\left(\sum_{i=1}^{(2+\epsilon)} \psi_i\psi_{ij}\right)^{2}\right)^{1/2}.$$

Therefore, by Cauchy-Schwartz applied to inner sum, we have

$$\left|\nabla|\nabla\psi|\right| = \frac{1}{|\nabla\psi|}\left(\sum_{i=1}^{(2+\epsilon)} \psi_i^2 \sum_{j=1}^{(2+\epsilon)}\sum_{i=1}^{(2+\epsilon)} \psi_{ij}^2\right)^{1/2} = \left(\sum_{i,j=1}^{(2+\epsilon)} \psi_i^2\right)^{1/2}$$

$$= |D^2\psi|.$$

**Remark 3.3.** The previous theorem could be considered as a form of the Pólya-Szegö principle for the Laplacian.A higher order version of Theorem 3.2 can now be obtained by iteration. The main problem here is to check that all requirements to implement the iteration are fulfilled.

**Theorem 3.4.** Let $\Omega$ be an open domain in $\mathbb{R}^{(2+\epsilon)}$, let $(1+\epsilon) \in N,\ \epsilon < \frac{1}{2}$, and let $Y(\Omega)$ be a r.i. space satisfying the $(P)$ and $Q(\epsilon)$-conditions. Then for all smooth functions $\varphi$ such that $|\nabla^m\varphi|^{**}(\infty) = 0, m = 0, 1, \dots, \epsilon$, the following inequality holds

$$\left\|(1-\epsilon)^{\left(\frac{-(1+\epsilon)}{2+\epsilon}\right)}\left(\varphi^{**}(1-\epsilon)-\varphi^{*}(1-\epsilon)\right)\right\|_{Y}$$

$$\lesssim \left\|\nabla^{(1+\epsilon)}\varphi\right\|_{Y}. \qquad (3.7)$$

Proof. Lemma 3.1 and Theorem 3.2 prove the cases $\epsilon = 0$, and $\epsilon = 1$. For the general case we now proceed by finite induction. Induction turns out to be the same for odd or even numbers $(1+\epsilon)$. Let $m$ be an arbitrary number such that $2 < m \leq 1 + \epsilon, m \equiv (1+\epsilon) \pmod 2$. Suppose that (3.7) is true for $m = 3 + \epsilon$. Suppose that $g$ is any function such that $\left\|\nabla^{(1+\epsilon)}\psi\right\|_Y < \infty$, and let $\varphi = \Delta\psi$. Applying (3.7) for $m = 3+\epsilon$, we get $\left\|(1-\epsilon)^{\left(\frac{-(m-2)}{2+\epsilon}\right)}\left(\Delta\psi^{**}(1-\epsilon) - \Delta\psi^{*}(1-\epsilon)\right)\right\|_Y \lesssim \|\nabla^{m-2}\Delta\psi\|_Y$.

Combining with Lemma 2.6 and the definition of $\nabla^m$ gives

$$\left\|(1-\epsilon)^{\left(\frac{-(m-2)}{2+\epsilon}\right)}(\Delta\psi)^{**}(1-\epsilon)\right\|_Y \lesssim \|\nabla^m\psi\|_Y. \quad (3.8)$$

Consider now the $r.i.$ space $B$ defined by the norm

$$\|h\|_B = \left\|(1-\epsilon)^{\left(\frac{-(m-2)}{2+\epsilon}\right)} h^{**}(1-\epsilon)\right\|_Y$$

By computation it is readily seen that $d_B(1+\epsilon) = d_Y(1+\epsilon)(1+\epsilon)^{\left(\frac{-(m-2)}{2+\epsilon}\right)}$. Therefore $B$ satisfies the $Q(1)$ -condition. Moreover, by Lemma 2.3, (2.3) holds in $B$ norm. Consequently we may now apply theorem 3.2 to derive

$$\left\|(1-\epsilon)^{\left(\frac{-2}{2+\epsilon}\right)}\left(\psi^{**}(1-\epsilon) - \psi^{*}(1-\epsilon)\right)\right\|_B \lesssim \|\Delta\psi\|_B$$

$$= \left\|(1-\epsilon)^{\left(\frac{-(m-2)}{2+\epsilon}\right)}(\Delta\psi)^{**}\right\|_Y \quad (3.9)$$

On the other hand,using the property $\|uv\|_Y \leq 2\|u^*v^*\|_Y$, which is valid for any r.i. space Y, we derive

$\|h\|_B \geq \left\|(1-\epsilon)^{\left(\frac{-(m-2)}{2+\epsilon}\right)} h^*(1-\epsilon)\right\|_Y \geq \frac{1}{2}\left\|(1-\epsilon)^{\left(\frac{-(m-2)}{2+\epsilon}\right)} h(1-\epsilon)\right\|_Y$ .Thus

$$\left\|(1-\epsilon)^{\left(\frac{-2}{2+\epsilon}\right)}\left(\psi^{**}(1-\epsilon)-\psi^*(1-\epsilon)\right)\right\|_B \geq \frac{1}{2}\left\|(1-\epsilon)^{\left(\frac{-m}{2+\epsilon}\right)}\left(\psi^{**}(1-\epsilon)-\psi^*(1-\epsilon)\right)\right\|_Y.$$

Combining the last estimate with (3.8) and (3.9) yields

$$\left\|(1-\epsilon)^{\left(\frac{-m}{2+\epsilon}\right)}\left(\psi^{**}(1-\epsilon)-\psi^*(1-\epsilon)\right)\right\|_Y \lesssim \|\nabla^m \psi\|_Y,$$

for any admissible $m$. It remains to but $m = 1+\epsilon$.

Example 3.5. Let $\Omega$ be an open domain in $\mathbb{R}^{(2+\epsilon)}$, and let $0 < -\epsilon^2+\epsilon < 2$.Then $L^{(2-\epsilon)}(\Omega)$ satisfies all the hypotheses of Theorem 3.4.

Proof. We give the details for the convenience of the reader. By Hardy's inequality we see that $L^{(2-\epsilon)}(\Omega)$ satisfies the $(P)$ - condition. Moreover since $d_Y(1+\epsilon) = (1+\epsilon)^{\left(\frac{1}{2-\epsilon}\right)}$ we see that for

$$0 < -\epsilon^2+\epsilon < 2 ,$$

$$Q(1+\epsilon, Y) = \int_1^\infty (1+\epsilon)^{\left(\frac{\epsilon}{2+\epsilon}\right)}(1+\epsilon)^{\left(\frac{-1}{2-\epsilon}\right)} \frac{d(1+\epsilon)}{(1+\epsilon)} < \infty$$

Proving that $L^{(2-\epsilon)}$ also satisfies the $Q(\epsilon)$ condition. In particular note that we may take $\epsilon = 0$ . We present the result which states that: if $X(\Omega)$ is a $r.i.$ space, $\epsilon < \frac{1}{2}$, then $\|\varphi\|_{X(\Omega)} \lesssim \left\|\nabla^{(1+\epsilon)}\varphi\right\|_{Y(\Omega)}$, for all $\varphi \in C_0^{1+\epsilon}(\Omega)$, implies that

$\left\|\int_{(1-\epsilon)}^{|\Omega|} \varphi(1+\epsilon)(1+\epsilon)^{\left(\frac{1+\epsilon}{2+\epsilon}\right)} \frac{d(1+\epsilon)}{1+\epsilon}\right\|_X \lesssim \|\varphi\|_Y$, for all $\varphi \geq 0$. ( 3.10)

As a consequence we prove.

**Theorem 3.6.** Suppose that $\epsilon < \frac{1}{2}$, $|\Omega| < \infty$, and $X(\Omega)$ is a $r.i.$ space such that $\|\varphi\|_{X(\Omega)} \lesssim \left\|\nabla^{(1+\epsilon)}\varphi\right\|_{Y(\Omega)}$, for all $\varphi \in C_0^\infty(\Omega)$.Then,

$$Y_{(2+\epsilon)}(\infty, 1+\epsilon)(\Omega) \subset X(\Omega), \qquad (3.11)$$

and moreover there exist a conatant $c = c(|\Omega|, X)$ such that $\|\varphi^*\|_{X(0,|\Omega|)} \leq c\|\varphi\|_{Y_{(2+\epsilon)}(\infty,1+\epsilon)}$.

**Proof.** By the fundamental theorem of calculus we have $\varphi^{**}(1-\epsilon) = \int_{(1-\epsilon)}^{\infty}\left(\varphi^{**}(1+\epsilon) - \varphi^*(1+\epsilon)\right)\frac{d(1+\epsilon)}{1+\epsilon} = \int_{(1-\epsilon)}^{|\Omega|}\left(\varphi^{**}(1+\epsilon) - \varphi^*(1+\epsilon)\right)\frac{d(1+\epsilon)}{1+\epsilon} + \int_{|\Omega|}^{\infty} \varphi^{**}(1+\epsilon)\frac{d(1+\epsilon)}{1+\epsilon} = \int_{(1-\epsilon)}^{|\Omega|}\left(\varphi^{**}(1+\epsilon) - \varphi^*(1+\epsilon)\right)\frac{d(1+\epsilon)}{1+\epsilon} + \frac{1}{|\Omega|}\int_0^{|\Omega|} \varphi^*(1+\epsilon)d(1+\epsilon)$.Therefore,

$$\begin{aligned}\|\varphi^*\|_{X(0,|\Omega|)} &\leq \|\varphi^{**}\|_{X(0,|\Omega|)} \\ &\leq \left\|\int_{(1-\epsilon)}^{|\Omega|}\left(\varphi^{**}(1+\epsilon)\right.\right. \\ &\quad \left.\left.- \varphi^*(1+\epsilon)\right)\frac{d(1+\epsilon)}{1+\epsilon}\right\|_{X(0,|\Omega|)} \\ &\quad + \frac{1}{|\Omega|}\int_0^{|\Omega|} \varphi^*(1+\epsilon)d(1+\epsilon)\|\chi_\Omega\|_{X(0,|\Omega|)}. \qquad (3.12)\end{aligned}$$

It remains to estimate the first term in the previous inequality Replacing $\varphi$ by $(1+\epsilon)^{\frac{-(1+\epsilon)}{2+\epsilon}}\left(\varphi^{**}(1+\epsilon)-\varphi^{*}(1+\epsilon)\right)$ in (3.10 we find

$$\left\|\int_{(1-\epsilon)}^{|\Omega|}\left(\varphi^{**}(1+\epsilon)-\varphi^{*}(1+\epsilon)\right)\frac{d(1+\epsilon)}{1+\epsilon}\right\|_{X}$$

$$\lesssim\left\|(1+\epsilon)^{\frac{-(1+\epsilon)}{2+\epsilon}}\left(\varphi^{**}(1+\epsilon)-\varphi^{*}(1+\epsilon)\right)\right\|_{Y}.$$

Therefore, inserting the last estimate in (3.12), we find

$\|\varphi^{*}\|_{X(0,|\Omega|)} \lesssim \|\varphi\|_{Y_{(2+\epsilon)}(\infty,1+\epsilon)} + \frac{1}{|\Omega|}\int_{0}^{|\Omega|}\varphi^{*}(1+\epsilon)d(1+\epsilon)\|\chi_{\Omega}\|_{X(0,|\Omega|)}$,and (3.11) follows.

## 4. Final Remarks

### 4.1. Equivalent conditions

In ([13] Maly and Pick, 2002) introduced the space $MP_{(2+\epsilon)}(\Omega) = \left\{\varphi: \varphi^*((1-\epsilon)/2) - \varphi^*(1-\epsilon) \in L^{(2+\epsilon)}(0,|\Omega|), \frac{d(1-\epsilon)}{1-\epsilon}\right\}$, and showed with ad-hoc methods that for $|\Omega| < \infty$,

$$W_0^{(1,2+\epsilon)}(\Omega) \subset MP_{(2+\epsilon)}(\Omega) \subsetneq H_{(2+\epsilon)}(\Omega) \qquad (4.1)$$

(see also [22] Kolyada,1989,Lemma 5.1). In ([11] Bastero, Milman and Ruiz) it was proved that

$$MP_{(2+\epsilon)}(\Omega) = L\big(\infty,(2+\epsilon)\big)(\Omega). \qquad (4.2)$$

For $\epsilon = 0$ the first half of Theorem 1.1. gives $W_0^{(1+\epsilon,2-\epsilon)}(\Omega) \subset L\left(\infty, \frac{2+\epsilon}{1+\epsilon}\right)(\Omega)$.

Therefore we could have used the second half of (4.1) combined with (4.2) and (1.5) to prove that the $L\left(\infty, \frac{2+\epsilon}{1+\epsilon}\right)$ spaces are optimal ([16] Milman and Pustylnik, 2004).

Likewise, since it is also shown independently in ([11] Bastero, Milman and Ruiz) that for $|\Omega| < \infty, \epsilon < 0, L(\infty, 1-\epsilon)(\Omega) \subset H_{(1-\epsilon)}(\Omega)$ ,we have still another method to prove that these spaces are optimal.In ([13] Maly and Pick, 2002) the proof of the fact that $MP_{(1-\epsilon)}(\Omega) \neq H_{(1-\epsilon)}(\Omega)$ (and therefore by (4.2) that $L(\infty, 1-\epsilon)(\Omega) \neq H_{(1-\epsilon)}(\Omega)$) is indirect.We show that $L(\infty, 1-\epsilon)(\Omega)$ is not a linear space.We prefer to give here a simple direct proof. To this end we now exhibit $\varphi \in H_{(1-\epsilon)}(\Omega)/L(\infty, 1-\epsilon)(\Omega)$ .

Let $\varphi(1-\epsilon) = \sum_{i=1}^{\infty} \chi_{(0,c_i)}(1-\epsilon)$, with $c_i = c^{-2^i}$. Then a computation shows that

$\varphi^{**}(1-\epsilon)-\varphi^{*}(1-\epsilon)=\frac{1}{(1-\epsilon)}\sum_{i=1}^{\infty} c_i\chi_{(c_i,\infty)}(1-\epsilon)$.There, for any $0>\epsilon>\infty$,

$$\int_0^1\left(\varphi^{**}(1-\epsilon)-\varphi^{*}(1-\epsilon)\right)^{(1-\epsilon)}\frac{d(1-\epsilon)}{1-\epsilon}$$

$$\geq\sum_{i=1}^{\infty}\int_0^1\frac{c_i^{(1-\epsilon)}\chi_{(c_i,\infty)}(1-\epsilon)}{(1-\epsilon)^{(1-\epsilon)}}\frac{d(1-\epsilon)}{1-\epsilon}$$

$$\geq\sum_{i=1}^{\infty}\int_{c_i}^1\frac{c_i^{(1-\epsilon)}}{(1-\epsilon)^{-\epsilon+2}}d(1-\epsilon)$$

$$=\frac{1}{1-\epsilon}\sum_{i=1}^{\infty}\left(1-c_i^{(1-\epsilon)}\right)=\infty.$$

On the other hand, by Hardy's inequality, $-i\left(\frac{(-\epsilon)}{1-\epsilon}\right)$

$$\|\varphi\|_{H_{(1-\epsilon)}}\approx\left(\int_0^1\left(\frac{\varphi^{*}(1-\epsilon)}{\ln\frac{e}{1-\epsilon}}\right)^{(1-\epsilon)}\frac{d(1-\epsilon)}{1-\epsilon}\right)^{\left(\frac{1}{1-\epsilon}\right)}$$

$$\leq\sum_{i=1}^{\infty}\left(\int_0^{c_i}\left(\ln\frac{1}{1-\epsilon}\right)^{-(1-\epsilon)}\frac{d(1-\epsilon)}{1-\epsilon}\right)^{\left(\frac{1}{1-\epsilon}\right)}$$

$$=-\frac{1}{\epsilon}\sum_{i=1}^{\infty}2^{-i\left(\frac{(-\epsilon)}{1-\epsilon}\right)}\leq\infty.$$

More generally, the method of the proof of (4.2) ginen in ([11] Bastero, Milman and Ruiz) can be easily extended to the spaces $Y_{(2+\epsilon)}(\infty,(1+\epsilon))$ introduced in this section so that we can readily show that

$$\|\varphi\|_{Y_{(2+\epsilon)}(\infty,1+\epsilon)} \approx \left\| (1-\epsilon)^{\left(\frac{-(1+\epsilon)}{2+\epsilon}\right)} \left( \varphi^* \left( \frac{1-\epsilon}{2} \right) - \varphi^*(1-\epsilon) \right) \right\|_Y .$$

It is interesting to observe that the conditions of our embedding theorems allow us to describe the $r.i.$ sets containing $W_0^{(1+\epsilon,2-\epsilon)}$ even for $(2-\epsilon)$ bigger than the critical value $\left(\frac{2+\epsilon}{1+\epsilon}\right)$. As is well known in this case $W_0^{(1+\epsilon,2-\epsilon)} \subset L^\infty$. Our rearrangement invariant sets are contained in $L^\infty$, which is a somewhat surprising property, since for finite measure spaces $L^\infty$ is the smallest rearrangement invariant space. It could be of interest to study this phenomena in detail. We now give an example of a result in this direction ([16] Milman and Pustylnik, 2004).

Example 4.1. Let $1 < -\epsilon^2 < 2 , \varphi \in W_0^{(1+\epsilon,2-\epsilon)}(\Omega)$.

Suppose that $\varphi^*\left(\frac{1-\epsilon}{2}\right) - \varphi^*(1-\epsilon)$ is a.e. equivalent to a monotone function on $(0,1)$. Then $\lim_{1-\epsilon\to 0} \frac{\varphi^*\left(\frac{1-\epsilon}{2}\right)-\varphi^*(1-\epsilon)}{(1-\epsilon)^{\frac{1+\epsilon}{(2+\epsilon)-\frac{1}{2-\epsilon}}}} = 0.$

The reason we were able to improve, in some cases, well known optimal embedding theorems, is the fact that the spaces $Y_{(2+\epsilon)}(\infty, 1+\epsilon)$ are only $r.i.$ sets. It is therefore of interest to ask if the $Y_{(2+\epsilon)}(\infty, 1+\epsilon)$ spaces are the optimal target for the Sobolev embedding theorem among all $r.i.$ sets.In particular, we conjecture that for $\epsilon = 0$ we have $r.i.$ hull

$$\left(W_0^{(1+\epsilon,2-\epsilon)}(\Omega)\right) = L\left(\infty, \frac{2+\epsilon}{1+\epsilon}\right)(\Omega).$$

Here the rearrangement invariant hull of

$$W_0^{(1+\epsilon,2-\epsilon)}(\Omega) = \left\{\varphi \in M(\Omega) : \exists\psi \in W_0^{(1+\epsilon,2-\epsilon)}(\Omega)\ (1+\epsilon)(1-\epsilon).\psi^* = \varphi^*\right\}.$$

In this respect we refer the reader to ([4] Bennett, DeVore and Sharpley, (1981)

where it is shown that for a cube $Q$, $r.i.$ hull $BMO(Q) = L(\infty,\infty)(Q)$.

It could also be of interest to consider the modified rearrangement invariant hull, as defined by ( [27] Netrusov,1989) modified $r.i.$ hull

$$\left(W_0^{(1+\epsilon,2-\epsilon)}(\Omega)\right) = \left\{\varphi : \exists\psi \in W_0^{(1+\epsilon,2-\epsilon)}(\Omega)\ (1+\epsilon).(1-\epsilon).\psi^* = \varphi^*\right\}.$$

A closely connected problem is to give necessary and sufficient conditions on $Y$ for the space $Y_{(2+\epsilon)}(\infty, 1+\epsilon)$ to be a $r.i.$ space. It follows from our results that a sufficient condition is for $Y$ to satisfy the $Q(1+\epsilon)$-condition. We conjecture that the condition is necessary as well. This conjecture is obiously true for $L(2-\epsilon, 1-\epsilon)$ spaces and can be readily verified for other spaces studied in ([15] Cwikel and Pustylnik,1998). Finally we would like to suggest that the $Y_{(2+\epsilon)}(\infty, 1+\epsilon)$ spaces introduced in this section should also be of interest in interpolation theory (cf.[26] Sagher and Shvartsman, (2002)

## Conclusion:

At last The main idea underlying our method is a suitable extension of the Pólya-Szegö symmetrization principle for higher order derivatives.It is easy to see that the standard form of this principle, comparing the first derivatives of a function $\varphi(x)$ and its non increasing rearrangent $\varphi^*(x)$ cannot be generalized even to second derivatives. Furthermore we verify the result that shows that our conditions are best possible.For this purpose we recall a result of ([6] Edmunds, Kerman and Pick,2000).We indicate some questions and open problems raised by our reaseach for this study.Finally we would like to suggest that the spaces introduced in this study should also be of interest in interpolation theory.

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