Sharp Embeddings of Besov Spaces Involving Small Logarithmic Smoothness

ShawgyHussein

Sudan University of Science and Technology, College of Science, Department of Mathematics, Sudan.

Hind Mohammed

Shaqra University, College of Science and Humanities in Shaqra, Department of Mathematics, Kingdom of Saudi Arabia.

Abstract

We use Kolyada' s inequality and its converse form and the work ofAntónio M. Caetano, Amiran Gogatishvili [17] to show sharp embeddings of Besov spaces $B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon}$ (involving the zero classical smoothness and a logarithmic smoothness with the smallexponent $\frac{\epsilon+\epsilon^2-1}{1+\epsilon}$) into Lorentz−Zygmund spaces. We also determine the small growth envelopes of spaces $B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon}$. In distinction to the case when theclassical smoothness is positive, we show that we cannot describe all embeddings in question in terms of the smallgrowth envelopes

Keywords: Besov spaces with generalized smoothness; Lorentz−Zygmund spaces; Sharp embeddings; Growth envelopes

استخدم كلوديس عدم المساواة وشكلها المعاكس وعمل انطو يونيو ام كايتانو واميران غوغايتشفيلي[17] لاثبات الغمر القاطع في فضاءات بيسوف

$B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon}$

تتضمن النعومه الكلاسيكية الصفرية والنعومه اللوغريثمية مع الاس الصغير

$\frac{\epsilon+\epsilon^2-1}{1+\epsilon}$)

في فضاءات لورانت وايضا حساب اصغر نمو لمغلفات الفضاءات

$B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon}$

للتغيير للحالة التي تكون فيها النعومه الكلاسيكية ايجابية بحيث نثبت انه لا يمكن وصف الغمر لاصغر نمو.

1. Introduction

We show sharp embeddings of Besovspaces $B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon} = B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon}(\mathbb{R}^n), 0 \le \epsilon < \infty$, into Lorentz–Zygmund spaces $L^{loc}_{1+\epsilon,1+3\epsilon;\gamma} = L^{loc}_{1+\epsilon,1+3\epsilon;\gamma}(\mathbb{R}^n), 0 \le \epsilon \le \infty$ and $\gamma \in \mathbb{R}$following the full methodology of António M. Caetano, Amiran Gogatishvili [17]. The Besovspaces $B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon}$, are defined by means of the modulus of continuity and they involve the zero classical smoothness and a logarithmic smoothness with the small exponent $\frac{\epsilon+\epsilon^2-1}{1+\epsilon}$—cf. By the Lorentz–Zygmund space $L^{loc}_{1+\epsilon,1+3\epsilon;\gamma}$ we mean the set of all measurable functions on $\mathbb{R}^n$ with the finite quasi-norm

$$\left(\int_0^1 (1+\epsilon)^{\frac{1+3\epsilon}{1+\epsilon}}(1+|\ln(1+\epsilon)|)^{\gamma(1+3\epsilon)} f^*(1+\epsilon)^{1+3\epsilon}\frac{d(1+\epsilon)}{1+\epsilon}\right)^{\frac{1}{1+3\epsilon}} \quad (1)$$

(with the usual modification when $\epsilon = \infty$).

First, Theorem 3.1 (see [17]) mentioned below states that the (continuous) embedding

$$B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon} \hookrightarrow L^{loc}_{1+\epsilon,1+3\epsilon;\gamma} \quad (2)$$

with

$$\gamma = \frac{\epsilon+3\epsilon^2-1}{1+3\epsilon} + \frac{1}{\max\{1+\epsilon,1+3\epsilon\}} \quad (3)$$

holds if and only if $\epsilon \ge 0$. Consequently, when $\epsilon \ge 0$, (2) holds with any γsatisfying

$$\gamma \le \frac{\epsilon+3\epsilon^2-1}{1+3\epsilon} + \frac{1}{\max\{1+\epsilon,1+3\epsilon\}}.$$

Second, if $\epsilon \ge 0$, then, by Theorem 3.2 mentioned below, embedding (2) cannot hold with $\gamma > \frac{\epsilon+3\epsilon^2-1}{1+3\epsilon} + \frac{1}{\max\{1+\epsilon,1+3\epsilon\}}$. This means that embedding (2) with γ given by (3) is sharp. Actually, Theorem 3.2 states even more. For example, it shows that we cannot make the target space in (2) (withγ from (3)) smaller by writing some small powers of iterated logarithms inside the quasi-norm (1) of the space $L^{loc}_{1+\epsilon,1+3\epsilon;\gamma}$ (see [17]).

There are two main ingredients of our proofs of these results. The first one is Kolyada's inequality recalled in Proposition 4.7. This inequality gives an estimate from below of the modulus of continuity of a function $f \in L_{1+\epsilon} = L_{1+\epsilon}(\mathbb{R}^n), 0 \le \epsilon < \infty$, in terms of its non-increasing rearrangement. The second one is the "inverse Kolyada inequality'' which is formulated in Proposition 3.5 and showed in this paper. Using these inequalities, we can reduce embedding (2) to a reverse Hardy inequality restricted to the cone of non-increasing functions—cf. Proposition 3.6 ([17]).

Embeddings of Besov spaces into rearrangement invariant spaces were considered by Goldman [7], Goldman and Kerman [8], and Netrusov [14]. These authors used different methods and considered a more general setting. However, as mentioned in [7], the characterization of embedding (2) can be obtained from [14] only when $\epsilon = 0$,here.Furthermore, the methods used in [7] also do not allow to consider the full range of parameters. Indeed, spaces was used. With this definition, the notion of the small growth envelope is meaningful even when $s = 0, 0 \leq \epsilon \leq \infty$ and $1 \leq 1+\epsilon \leq \min\{1+\epsilon, 2\}$ (a so-called borderline case). The best what is known in such a case—cf. [12]—is that the growth envelope function is $(1+\epsilon)^{-\frac{1}{1+\epsilon}}$ (as expected), and that the fine index should be between $1+\epsilon$.

Growth envelopes have been also studied for Besov spaces $B^{(s,\Psi)}_{1+\epsilon,1+\epsilon}$ in [4,3,9], where Ψ stands for a function of log-type and $s \in (0, \frac{n}{1+\epsilon}]$. We refer to [2,10,1] for results on growth envelopes of more general Besov (and also Triebel–Lizorkin) spaces of generalized smoothness. While in [2–4] the Fourier analytical definition of spaces was used, in [9,10] an equivalent definition based on the modulus of smoothness was employed.
On the other hand, no information has been obtained for the borderline case mentioned above when $s = 0$ and when all the known techniques do not work.

We determine the smallgrowth envelope of the Besovspace $B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon}$, (that is when $s = 0$) defined by means of the modulus of continuity that proved by [17]. If $0 \leq \epsilon < \infty$, then the smallgrowth envelope of the space $B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon}$, is the pair $((1+\epsilon)^{-\frac{1}{1+\epsilon}}(1+|\ln(1+\epsilon)|)^{-\epsilon}, \max\{1+\epsilon, 1+\epsilon\})$—cf. Theorem 3.3. There are some interesting features of this result. In distinction to results on smallgrowth envelopes of Besov spaces $B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon}$ with $s \in (0, n/(1+\epsilon)]$, the first index $1+\epsilon$ plays a new role here: it is involved in the fine index, which is not $1+\epsilon$ now but $\max\{1+\epsilon, 1+\epsilon\}$. Furthermore, another new phenomenon appears here. Namely, the embedding of the Besov space $B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon}$,given by Theorem 3.1 cannot be described in terms of the smallgrowth envelope of the space $B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+3\epsilon,1+\epsilon}$, when $0 \leq \epsilon < \infty$—cf. $1+3\epsilon, 1+\epsilon$ −fc.Remark 3.4. The paper is organized as follows. In Section 2 we give notation and basic definitions. Main results are presented in Section 3. Section 4 is devoted to auxiliary assertions. In subsequent sections (Sections 5–9) main results are shown.

2. Notation and basic definitions

Given two non-negative expressions $\mathcal{A}$and $\mathcal{B}$, the symbol $\mathcal{A} \lesssim \mathcal{B}$ means that $\mathcal{A} \lesssim c\mathcal{B}$ for some positive constant c independent of the variables in the expressions $\mathcal{A}$ and $\mathcal{B}$. If $\mathcal{A} \lesssim \mathcal{B}$and $\mathcal{B} \lesssim \mathcal{A}$, we write $\mathcal{A} \approx \mathcal{B}$ and say that $\mathcal{A}$ and $\mathcal{B}$ are equivalent.
Given a set A, its characteristic function is denoted by χ_A. Given two sets $A$and B, we write $A\Delta B$ for their symmetric difference. For $a \in \mathbb{R}^n$and $\epsilon \geq 0$, the notation $B(a, 1+\epsilon)$ stands for the small closed ball in $\mathbb{R}^n$ centred at a with radius $1+\epsilon$. The volume of $B(0,1)$ in $\mathbb{R}^n$ is denoted by V_n though, in general, we use the notation $|\cdot|_n$ for Lebesgue measure in $\mathbb{R}^n$.

Let Ω be a Borel subset of $\mathbb{R}^n$. The symbol $\mathcal{M}_0(\Omega)$ is used to denote the family of all complex-valued or extended real-valued (Lebesgue-)measurable functions defined and finite a.e. on Ω. By $\mathcal{M}_0^+(\Omega)$ we mean the subset of $\mathcal{M}_0(\Omega)$ consisting of those functions which are non-negative a.e. on Ω. If $\Omega = (a, a+\epsilon) \subset \mathbb{R}$, we write simply $\mathcal{M}_0(a, a+\epsilon)$ and $\mathcal{M}_0^+(a, a+\epsilon)$ instead of $\mathcal{M}_0((a, a+\epsilon))$ and $\mathcal{M}_0((a, a+\epsilon))$, respectively. By $\mathcal{M}_0^+(a, a+\epsilon; \downarrow)$ or $\mathcal{M}_0^+(a, a+\epsilon; \uparrow)$ we mean the collection of all $f \in \mathcal{M}_0^+(a, a+\epsilon)$ which are non-increasing or non-decreasing on $(a, a+\epsilon)$, respectively. Finally, by $AC(a, a+\epsilon)$ we denote the family of all real-valued functions which are locally absolutely continuous on $(a, a+\epsilon)$ (that is, absolutely continuous on any small closed subinterval of is finite. When $\Omega = \mathbb{R}^n$, we simplify $L_{1+\epsilon}(\Omega)$ to $L_{1+\epsilon}$ and $\|\cdot\|_{1+\epsilon,\Omega}$, to $\|\cdot\|_{1+\epsilon}$.

Given $f \in L_{1+\epsilon}$, $0 \le \epsilon < \infty$, the first difference operator Δ_h of step $h \in \mathbb{R}^n$ transforms f in $\Delta_h f$ defined by

$$(\Delta_h f)(x^2) := f(x^2 + h) - f(x^2), \qquad x^2 \in \mathbb{R}^n,$$

whereas the modulus of continuity of f is given by

$$\omega_1^2(f, 1+\epsilon)_{1+\epsilon} := \sup_{\substack{h \in \mathbb{R}^n \\ |h| \le 1+\epsilon}} \|\Delta_h f\|_{1+\epsilon}, \qquad \epsilon \ge 0.$$

We introduce (see [17]) the Besov function spaces with the zero classical smoothness which we shall consider. The smoothness will be controlled by somesmall power of $\ell(1+\epsilon)$, where $\ell(1+\epsilon) := 1 + |\ln 1 + \epsilon|$, $\epsilon \ge 0$.

Definition 2.1. Given $0 \le \epsilon < \infty$,and $\frac{\epsilon+\epsilon^2-1}{1+\epsilon} \in \mathbb{R}$,

$$B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon} := \left\{ f \in L_{1+\epsilon} : \|f\|_{B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon}} := \|f\|_{1+\epsilon} + \left\| (1+\epsilon)^{-\frac{1}{1+\epsilon}} \ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}(1+\epsilon)\, \omega_1^2(f, 1+\epsilon)_{1+\epsilon} \right\|_{1+\epsilon,(0,1)} < \infty \right\}.$$

Note that, since $\omega_1^2(f, 1+\epsilon)_{1+\epsilon} \lesssim \|f\|_{1+\epsilon}$, only the case $\left\| (1+\epsilon)^{-\frac{1}{1+\epsilon}} \ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}(1+\epsilon) \right\|_{1+\epsilon,(0,1)} = \infty$ (or, equivalently, $\epsilon \ge 0$ if $1+\epsilon$ is finite and $\epsilon > 0$ if $1+\epsilon$ is infinity) is of interest; otherwise $B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon} = L_{1+\epsilon}$.

We shall occasionally need the notion of Borel measure μ associated with a non-decreasing function $g : (a, a+\epsilon) \in \mathbb{R}$, where $-\infty \le \epsilon \le \infty$. By this we mean the unique (non-negative) measure on the Borel subsets of $(a, a+\epsilon)$ such that $\mu([c,d]) = g(d+) - g(c-)$ for all $[c,d] \subset (a, a+\epsilon)$.

We show the notion of growth envelope of the function space A (see [12] for details).

Definition 2.2. Let $(A, \|\cdot\|_A) \subset \mathcal{M}_0(\mathbb{R}^n)$ be a quasi-normed space such that $A \nrightarrow L_\infty$. A positive, non-increasing, continuous function μ_h defined on some interval $(0, \varepsilon]$, $\varepsilon \in (0,1)$, is called the (local) growth envelope function of the space A provided that

$$h(1+\epsilon) \approx \sup_{\|f\|_A \le 1} f^*(1+\epsilon) \quad \text{for all} -1 < \epsilon \le \varepsilon - 1.$$

Given a growth envelope function h of the space A (determined up to equivalence near zero) and a number $u \in (0, \infty]$, we call the pair (h, u) the (local) growth envelope of the space A when the inequality

$$\left(\int_{(0,\varepsilon)} \left(\frac{f^*(1+\epsilon)}{h(1+\epsilon)} \right) d\mu_H(1+\epsilon) \right)^{\frac{1}{1+2\epsilon}} \lesssim \|f\|_A$$

(with the usual modification when $\epsilon = \infty$) holds for all $f \in A$ if and only if the positive exponent $(1+2\epsilon)$ satisfies $1 + 2\epsilon \ge u$. Here μ_H is the Borel measure associated with the non-decreasing function $H(1+\epsilon) := -\ln h(1+\epsilon)$, $\epsilon > 0$. The component u in the growth envelope pair is called the fine index.

3. Main results (see [17])

Theorem 3.1. If $0 \le \epsilon < \infty$, then the inequality

$$\left\| (1+\epsilon)^{\frac{\epsilon}{(1+\epsilon)(1+2\epsilon)}} \ell^{\frac{1}{\max\{1+\epsilon, 1+2\epsilon\}} - \frac{\epsilon(1+2\epsilon)-1}{1+2\epsilon}}(1+\epsilon) f^*(1+\epsilon) \right\|_{1+2\epsilon,(0,1)} \lesssim \|f\|_{B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon}} \tag{4}$$

holds for all $f \in B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon}$, if and only if $\epsilon \ge 0$.

Theorem 3.2. Let $0 \le \epsilon < \infty$, and let $\kappa \in \mathcal{M}_0^+(0,1;\downarrow)$. Then the inequality

$$\left\| (1+\epsilon)^{\frac{\epsilon}{(1+\epsilon)(1+2\epsilon)}} \ell^{\frac{\epsilon+2\epsilon^2-1}{1+2\epsilon} + \frac{1}{\max\{1+\epsilon,1+2\epsilon\}}}(1+\epsilon)\, \kappa(1+\epsilon) f^*(1+\epsilon) \right\|_{1+2\epsilon,(0,1)} \lesssim \|f\|_{B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon}} \tag{5}$$

holds for all $f \in B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon}$, if and only if κ is bounded.

is small enough. Since $H'(1+\epsilon) \approx \frac{1}{1+\epsilon}$ for all $\epsilon > 0$, the measure μ_H associated with the function H satisfies $d\mu_H(1+\epsilon) \approx \frac{d(1+\epsilon)}{1+\epsilon}$. Thus, by Definition 2.2 and Theorem 3.3,

$$\left\| (1+\epsilon)^{-\frac{1}{1+2\epsilon}} \frac{f^*(1+\epsilon)}{h(1+\epsilon)} \right\|_{1+2\epsilon,(0,\varepsilon)} \lesssim \|f\|_{B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon}} \text{for all } f \in B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon} \tag{6}$$

provided that

$$1+2\epsilon \geq \max\{1+\epsilon, 1+\epsilon\}. \tag{7}$$

Hence, if (7) holds, then inequality (6) gives the same result as inequality (4) of Theorem 3.1. However, if $\epsilon > 0$, inequality (6) does not hold (cf. Theorem 3.2), while inequality (4) does. This means that the embeddings of Besov spaces $B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+3\epsilon,1+\epsilon}$, given by Theorem 3.1 cannot be described in terms of growth envelopes when $0 \leq \epsilon < \infty$.

Two of the main ingredients in the proofs of Theorems 3.1–3.3 are Proposition 4.7 (Kolyada's inequality) and Proposition 3.5 (which we call the "inverse'' Kolyada inequality) mentioned below.

Proposition 3.5. (i) Let $f \in L_1$ and let $F(x^2) := f^*(V_n|x^2|^n), x^2 \in \mathbb{R}^n$. Then

$$\omega_1^2(F, 1+\epsilon)_1 \lesssim n \int_0^{(1+\epsilon)^n} f^*(s)ds + (n-1)(1+\epsilon) \int_{(1+\epsilon)^n}^{\infty} f^*(s)s^{-\frac{1}{n}}ds$$
$$= (1+\epsilon) \left(\int_{(1+\epsilon)^n}^{\infty} s^{-1/n} \int_0^s \left(f^*(u) - f^*(s)\right) du \frac{ds}{s} \right) \tag{8}$$

for all $\epsilon \geq 0$ and $f \in L_1$.

(ii) Let $0 < \epsilon < \infty, f \in L_{1+\epsilon}$ and let $F(x^2) = f^{**}(V_n|x^2|^n), x^2 \in \mathbb{R}^n$. Then

$$\omega_1^2(F, 1+\epsilon)_{1+\epsilon} \lesssim (1+\epsilon) \left(\int_{(1+\epsilon)^n}^{\infty} s^{-\frac{1+\epsilon}{n}} \int_0^s \left(f^*(u) - f^*(s)\right)^{1+\epsilon} du \frac{ds}{s} \right)^{\frac{1}{1+\epsilon}}$$

for all $\epsilon \geq 0$ and $f \in L_{1+\epsilon}$.

In fact, Propositions 4.7 and 3.5 enable us to reduce the embedding in question to the following assertion:

Proposition 3.6. Let $0 \leq \epsilon < \infty, \frac{\epsilon+\epsilon^2-1}{1+\epsilon} \in \mathbb{R}$ and let ω^2 be a measurable function on $(0,1)$. Then

$$\|\omega^2(1+\epsilon) f^*(1+\epsilon)\|_{1+2\epsilon,(0,1)} \lesssim \|f\|_{B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon}} \tag{9}$$

for all $f \in B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon}$, if and only if

$\|\omega^2(1+\epsilon) f^*(1+\epsilon)\|_{1+2\epsilon,(0,1)}$

$$\lesssim \left\| (1+\epsilon)^{\frac{\epsilon}{1+\epsilon}} \ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}} (1+\epsilon) \left(\int_{(1+\epsilon)^n}^{2} s^{-\frac{1+\epsilon}{n}} \int_0^s \left(f^*(u) - f^*(s)\right)^{1+\epsilon} du \frac{ds}{s} \right)^{\frac{1}{1+\epsilon}} \right\|_{1+\epsilon,(0,1)} \tag{10}$$

for all $f \in \mathcal{M}_0(\mathbb{R}^n)$ such that $|\mathrm{supp}\, f|_n \leq 1$.

4. Preliminaries

The following easy estimates are quite useful and will be used without further notice whenever convenient: if $0 < \epsilon \leq \infty$and $(a+\epsilon) \in \mathbb{R}$, then

$$\left\| (1+\epsilon)^{\frac{\epsilon(1+\epsilon)-1}{1+\epsilon}} \ell(1+\epsilon)^{a+\epsilon} \right\|_{1+\epsilon,(0,T)} \approx T^{\varepsilon} \ell(T)^{a+\epsilon} \text{and} \left\| (1+\epsilon)^{-\frac{\epsilon(1+\epsilon)+1}{1+\epsilon}} \ell(1+\epsilon)^{a+\epsilon} \right\|_{1+\epsilon,(T,\infty)} \approx T^{\varepsilon} \ell(T)^{a+\epsilon}$$

for all $T \in (0, \infty)$.

We shall also need the following geometric estimate (see [17]):

Proposition 4.1. For all $a, a+\epsilon \in \mathbb{R}^n$and $\epsilon \geq 0$,

$$|B(a, 1+\epsilon) \Delta B(a+\epsilon, 1+\epsilon)|_n \lesssim |\epsilon|(1+\epsilon)^{n-1}. \tag{11}$$

Proof. Since the cases $\epsilon = 0$ or $\epsilon = -1$ are obvious, we assume that $\epsilon = 0$and $\epsilon > 0$.

If $|\epsilon| > \frac{1+\epsilon}{2}$, then $|B(a, 1+\epsilon) \Delta B(a+\epsilon, 1+\epsilon)|_n \lesssim (1+\epsilon)^n < 2|\epsilon|(1+\epsilon)^{n-1}$ and (11) follows.

If $|\epsilon| \leq \frac{1+\epsilon}{2}$, then the inclusion $B(a, 1+\epsilon - |\epsilon|) \subset B(a+\epsilon, 1+\epsilon)$ and its symmetric counterpart $B(a+\epsilon, 1+\epsilon - |\epsilon|) \subset B(a, 1+\epsilon)$ imply that

$$B(a, 1+\epsilon) \Delta B(a+\epsilon, 1+\epsilon) \subset \left(B(a, 1+\epsilon) \backslash B(a, 1+\epsilon - |\epsilon|)\right) \cup \left(B(a+\epsilon, 1+\epsilon) \backslash B(a+\epsilon, 1+\epsilon - |\epsilon|)\right).$$

Consequently,

$$|B(a, 1+\epsilon) \Delta B(a+\epsilon, 1+\epsilon)|_n \lesssim (1+\epsilon)^n - (1+\epsilon - |\epsilon|)^n,$$

$$\leq |\epsilon|(1+\epsilon)^{n-1}\left(n+\sum_{j=2}^{n}\binom{n}{j}2^{-(j-1)}\right)\approx |\epsilon|(1+\epsilon)^{n-1}.$$

Next we present two monotonicity results (see [17]):

Proposition 4.2. Given $\epsilon \geq 0$ and a non-increasing function $g:(0,\infty)\to\mathbb{R}$, the function

$$1+\epsilon \mapsto \int_0^{1+\epsilon}\left(g(s)-g(1+\epsilon)\right)^{1+\epsilon}ds \tag{12}$$

is non-decreasing on $(0,\infty)$. In particular, if $f\in\mathcal{M}_0(\mathbb{R}^n)$, then the functions

$$1+\epsilon \mapsto \int_0^{1+\epsilon}\left(f^*(s)-f^*(1+\epsilon)\right)^{1+\epsilon}ds \tag{13}$$

and

$$1+\epsilon \mapsto (1+\epsilon)\left(f^{**}(1+\epsilon)-f^*(1+\epsilon)\right) \tag{14}$$

are non-decreasing on $(0,\infty)$.

Proof. Given $0<\epsilon<\infty$,

$$\int_0^{1+\epsilon}\left(g(s)-g(1+\epsilon)\right)^{1+\epsilon}ds \leq \int_0^{1+\epsilon}\left(g(s)-g(1+2\epsilon)\right)^{1+\epsilon}ds \leq \int_0^{1+2\epsilon}\left(g(s)-g(1+2\epsilon)\right)^{1+\epsilon}ds.$$

Proposition 4.3. Let be a (non-negative) measure on $(0,\infty)$ such that $[1+\epsilon,\infty)\in(0,\infty)$ for all $0\leq\epsilon<\infty$. Let $g\in\mathcal{M}_0^+(0,\infty;\uparrow)$. Then the function

$$1+\epsilon\mapsto \mu[1+\epsilon,\infty)^{-1}\int_{[1+\epsilon,\infty)}g\,d\mu$$

is also non-decreasing on $(0,\infty)$.

Proof. First note that the conclusion is plain if $\int_{[1+\epsilon,\infty)}g\,d\mu$ is infinite for all $(1+\epsilon)$. On the other hand, if it is finite for some $(1+\epsilon)$, it is finite for all $(1+\epsilon)$ (due to the hypotheses of the proposition). Therefore,for $0<\epsilon<\infty$,

$$\frac{1}{\mu[1+\epsilon,\infty)}\int_{[1+\epsilon,\infty)}g\,d\mu=\frac{\int_{[1+\epsilon,1+2\epsilon)}g\,d\mu+\int_{[1+2\epsilon,\infty)}g\,d\mu}{\mu[1+\epsilon,\infty)}\leq\frac{\mu[1+\epsilon,1+2\epsilon)g(1+2\epsilon)+\int_{[1+2\epsilon,\infty)}g\,d\mu}{\mu[1+\epsilon,\infty)}$$

$$=\frac{\mu[1+\epsilon,1+2\epsilon)\mu[1+2\epsilon,\infty)^{-1}\mu[1+2\epsilon,\infty)g(1+2\epsilon)+\int_{[1+2\epsilon,\infty)}g\,d\mu}{\mu[1+\epsilon,\infty)}$$

$$\leq\frac{\mu[1+\epsilon,1+2\epsilon)\mu[1+2\epsilon,\infty)^{-1}\int_{[1+2\epsilon,\infty)}g\,d\mu+\int_{[1+2\epsilon,\infty)}g\,d\mu}{\mu[1+\epsilon,\infty)}$$

$$=\frac{\mu[1+\epsilon,1+2\epsilon)+\mu[1+2\epsilon,\infty)}{\mu[1+\epsilon,\infty)\mu[1+2\epsilon,\infty)}\int_{[1+2\epsilon,\infty)}g\,d\mu=\frac{1}{\mu[1+2\epsilon,\infty)}\int_{[1+2\epsilon,\infty)}g\,d\mu.$$

Now we proceed by recalling some properties of the maximal functions f^{**} of elements $f\in L_{1+\epsilon},0\leq\epsilon\leq\infty$. Such functions f are locally integrable in $\mathbb{R}^n$ and so the function $1+\epsilon\mapsto\int_0^{1+\epsilon}f^*(s)ds$ belongs to $AC(0,\infty)$ and

$$\frac{d}{d(1+\epsilon)}\int_0^{1+\epsilon}f^*(s)ds=f^*(1+\epsilon)\quad a.e.\,in\,(0,\infty).$$

Consequently,

$$(f^{**})'(1+\epsilon)=-\frac{1}{1+\epsilon}\left(f^{**}(1+\epsilon)-f^*(1+\epsilon)\right)\quad a.e.\,in\,(0,\infty). \tag{15}$$

On the other hand, since the function $1+\epsilon\mapsto\frac{1}{1+\epsilon}$ also belongs to $AC(0,\infty)$, the same can be said about f^{**} and we can write, for any $0<\epsilon<\infty$,

$$f^{**}(1+2\epsilon)-f^{**}(1+\epsilon)=\int_{1+\epsilon}^{1+2\epsilon}(f^{**})'(s)ds=\int_{1+\epsilon}^{1+2\epsilon}\left((f^{**})(s)-f^*(s)\right)ds \tag{16}$$

In order to prove the next proposition involving $f^*$and f^{**}, we need classical Hardy's inequalities (see, for example, [11, pp. 240, 244]) and [17].

Given $0<\epsilon<\infty$ and a non-negative, measurable function f on $(0,\infty)$, and that

$$\int_{[0,\infty]} \frac{v^2(s)}{s^{1+2\epsilon} + (1+\epsilon)^{1+\epsilon}} ds < \infty$$

for all $0 \le \epsilon < \infty$. Then the inequality

$$\left(\int_0^\infty w^2(1+\epsilon) f^*(1+\epsilon)^{1+\epsilon} d(1+\epsilon)\right)^{\frac{1}{1+\epsilon}} \lesssim \left(\int_0^\infty v^2(1+\epsilon) f^{**}(1+\epsilon)^{1+2\epsilon} d(1+\epsilon)\right)^{\frac{1}{1+2\epsilon}} \quad (19)$$

holds for all measurable f on $\mathbb{R}^n$ if and only if

$$\int_0^\infty \frac{(1+\epsilon)^{\frac{(1+2\epsilon)(1+\epsilon)}{\epsilon}} \sup_{y^2\in[1+\epsilon,\infty)} y^{-\frac{2(1+2\epsilon)(1+\epsilon)}{\epsilon}} W(y^2)^{\frac{1+2\epsilon}{\epsilon}}}{\left(V(1+\epsilon) + (1+\epsilon)^{1+2\epsilon}\int_{1+\epsilon}^\infty s^{-(1+2\epsilon)} v^2(s) ds\right)^{\frac{1+3\epsilon}{\epsilon}}} V(1+\epsilon) \int_{1+\epsilon}^\infty s^{-(1+2\epsilon)} v^2(s) ds\, (1+\epsilon)^{2\epsilon} d(1+\epsilon) < \infty. \quad (20)$$

Proposition 4.9. Let $0 \le \epsilon < \infty$, let v^2, w^2 be non-negative, locally integrable functions on $(0,\infty)$ and put $W(1+\epsilon) := \int_0^{1+\epsilon} w^2(s) ds, \epsilon \ge 0$. Consider the function

$$\varphi(1+\epsilon) := \operatorname*{ess\,sup}_{s\in(0,1+\epsilon)} s \operatorname*{ess\,sup}_{\tau\in(0,\infty)} \frac{v^2(\tau)}{\tau}, \quad 0 \le \epsilon < \infty. \quad (21)$$

This function is quasi-concave (that is, φ is equivalent to a function in $\mathcal{M}_0^+(0,\infty;\uparrow)$ while $\frac{\varphi(1+\epsilon)}{1+\epsilon}$ is equivalent to a function in $\mathcal{M}_0^+(0,\infty;\downarrow)$. Assume that φ is non-degenerate, that is,

$$\lim_{1+\epsilon\to 0_+} \varphi(1+\epsilon) = \lim_{\epsilon\to\infty} \frac{1}{\varphi(1+\epsilon)} = \lim_{\epsilon\to\infty} \frac{\varphi(1+\epsilon)}{1+\epsilon} = \lim_{1+\epsilon\to 0_+} \frac{1+\epsilon}{\varphi(1+\epsilon)} = 0. \quad (22)$$

Let v^2 be a non-negative Borel measure on $[0,\infty)$ such that

$$\frac{1}{\varphi(1+\epsilon)^{1+\epsilon}} \approx \int_{[0,\infty)} \frac{dv^2(s)}{s^{1+\epsilon} + (1+\epsilon)^{1+\epsilon}} \text{ for all } 0 \le \epsilon < \infty.$$

Then the inequality

$$\left(\int_0^\infty w^2(1+\epsilon) f^*(1+\epsilon)^{1+\epsilon} d(1+\epsilon)\right)^{\frac{1}{1+\epsilon}} \lesssim \operatorname*{ess\,sup}_{0\le\epsilon<\infty} v^2(1+\epsilon) f^{**}(1+\epsilon) \quad (23)$$

holds for all measurable f on $\mathbb{R}^n$ if and only if

$$\int_{[0,\infty)} \sup_{s\in(1+\epsilon,\infty)} \frac{W(s)}{s^{1+\epsilon}} dv^2(1+\epsilon) < \infty. \quad (24)$$

5. Proof of Proposition 3.5

First we prove the following auxiliary result [17]:

Lemma 5.1. Let $g \in \mathcal{M}_0^+(0,\infty;\downarrow)$ and let $F(x^2) := g(V_n|x^2|^n), x^2 \in \mathbb{R}^n$. Then

$$\|\Delta_h F\|_1 \lesssim n \int_0^{V_n|h|^n} g(s) ds + (n-1) V_n^{1/n} |h| \int_{V_n|h|^n}^\infty g(s) s^{-1/n} ds \quad (25)$$

for all $h \in \mathbb{R}^n \setminus \{0\}$ and $g \in \mathcal{M}_0^+(0,\infty;\downarrow)$.

Moreover, if $g \in AC(0,\infty)$and $0 \le \epsilon < \infty$, then

$$\begin{aligned}\|\Delta_h F\|_{1+\epsilon} &\lesssim n\left(\int_0^{V_n 3^n|h|^n} \left(g(s) - g(V_n 3^n |h|^n)\right)^{1+\epsilon} ds\right)^{\frac{1}{1+\epsilon}} \\ &+ |h|\left(\int_{V_n 2^n|h|^n}^\infty s^{\left(1-\frac{1}{n}\right)(1+\epsilon)} \operatorname*{ess\,sup}_{s/2^n\le u\le 3^n s/2^n} |g'(u)|^{1+\epsilon} ds\right)^{\frac{1}{1+\epsilon}} \quad (26)\end{aligned}$$

for all $h \in \mathbb{R}^n \setminus \{0\}$ and $g \in \mathcal{M}_0^+(0,\infty;\downarrow) \cap AC(0,\infty)$.

Proof.

Step 1: Assume that $g \in \mathcal{M}_0^+(0,\infty;\downarrow)$. Then

$$\|\Delta_h F\|_1 = \int_{|x^2|<2|h|} |F(x^2+h) - F(x^2)| dx^2 + \int_{|x^2|>2|h|} |F(x^2+h) - F(x^2)| dx^2 =: I + II.$$

Using polar coordinates, the definition of F and a further change of variables, we obtain

$$\lesssim \left(\int_{|x^2|>2|h|} |h|^{1+\epsilon}|x^2|^{(n-1)(1+\epsilon)} \operatorname*{ess\,sup}_{V_n|x^2|^n/2^n \le u \le V_n 3^n|x^2|^n/2^n} |g'(u)|^{1+\epsilon}\, dx^2\right)^{\frac{1}{1+\epsilon}}$$

$$\approx |h|\left(\int_{2|h|}^{\infty} (1+\epsilon)^{(n-1)(1+\epsilon)} \operatorname*{ess\,sup}_{V_n(1+\epsilon)^n/2^n \le u \le V_n 3^n(1+\epsilon)^n/2^n} |g'(u)|^{1+\epsilon}\,(1+\epsilon)^{n-1} d(1+\epsilon)\right)^{\frac{1}{1+\epsilon}}$$

$$\approx |h|\left(\int_{V_n 2^n|h|^n}^{\infty} s^{\left(1-\frac{1}{n}\right)(1+\epsilon)} \operatorname*{ess\,sup}_{s/2^n \le u \le 3^n s/2^n} |g'(u)|^{1+\epsilon}\, ds\right)^{\frac{1}{1+\epsilon}}.$$

Together with (30) and (29), this yields (26).

Proof of Proposition 3.5.

Step 1: To prove (i), take $f \in L_1$ and $g = f^*$ in Lemma 5.1. Consequently,

$$\|\Delta_h F\|_1 \lesssim n\int_0^{V_n|h|^n} f^*(s)ds + (n-1)V_n^{1/n}|h|\int_{V_n|h|^n}^{\infty} f^*(s)s^{-1/n}ds. \tag{31}$$

Applying Fubini's theorem and the fact that f^* is integrable on $(0,\infty)$, we can rewrite the last expression as

$$V_n^{1/n}|h|\left(\int_{V_n|h|^n}^{\infty} s^{-1/n}\int_0^s \left(f^*(u)-f^*(s)\right)du\frac{ds}{s}\right)$$

$$= V_n^{1/n}|h|\left(\int_{V_n|h|^n}^{\infty} s^{-1/n-1}\int_0^s f^*(u)du\, ds\right) - V_n^{1/n}|h|\left(\int_{V_n|h|^n}^{\infty} s^{-1/n-1}\int_0^s f^*(s)du\, ds\right)$$

$$= n\int_0^{V_n|h|^n} f^*(u)du + nV_n^{1/n}|h|\int_{V_n|h|^n}^{\infty} f^*(u)u^{-1/n}du - V_n^{1/n}|h|\int_{V_n|h|^n}^{\infty} f^*(s)s^{-1/n}ds$$

$$= n\int_0^{V_n|h|^n} f^*(u)du + (n-1)V_n^{1/n}|h|\int_{V_n|h|^n}^{\infty} f^*(u)u^{-1/n}du\,. \tag{32}$$

When $n = 1$, it is plain that the right-hand side of (31) is non-decreasing in $|h|$. When $n > 1$, it is also non-decreasing in $|h|$, which can be seen from the equivalent expression given in (32) and from Proposition 4.3 (with $d\mu(s) = s^{-1/n-1}ds$ and $g(s) = n\int_0^s \left(f^*(u)-f^*(s)\right)du$; the fact that $g \in \mathcal{M}_0^+(0,\infty;\uparrow)$follows from Proposition 4.2).

Now, (31) and (32) imply that

$$\omega_1^2(F,1+\epsilon)_1 \lesssim n\int_0^{V_n(1+\epsilon)^n} f^*(s)ds + (n-1)V_n^{1/n}(1+\epsilon)\int_{V_n(1+\epsilon)^n}^{\infty} f^*(s)s^{-1/n}ds$$

$$= V_n^{1/n}(1+\epsilon)\left(\int_{V_n(1+\epsilon)^n}^{\infty} s^{-1/n}\int_0^s \left(f^*(u)-f^*(s)\right)du\frac{ds}{s}\right).$$

In order to complete the proof of Proposition 3.5(i), note that the factors V_n and $V_n^{1/n}$ can be omitted in the preceding formulae (this follows again by arguments used in (32) and the discussion following it).

Step 2: To prove part (ii), take $f \in L_{1+\epsilon}, 0 < \epsilon < \infty$, and $g = f^{**}$ in Lemma 5.1. Consequently,

$$\|\Delta_h F\|_{1+\epsilon}$$

$$\lesssim \left(\int_0^{V_n 3^n|h|^n} \left(f^{**}(s) - f^{**}(V_n 3^n|h|^n)\right)^{1+\epsilon} ds\right)^{\frac{1}{1+\epsilon}}$$

$$+ |h|\left(\int_{V_n 2^n|h|^n}^{\infty} s^{\left(1-\frac{1}{n}\right)(1+\epsilon)} \operatorname*{ess\,sup}_{s/2^n \le u \le 3^n s/2^n} |(f^{**})'(u)|^{1+\epsilon}\, ds\right)^{\frac{1}{1+\epsilon}} \tag{33}$$

since $\in L_{1+\epsilon}$, (15) yields

$$(f^{**})'(u) = -\frac{1}{u}\left((f^{**})'(u) - (f^*)'(u)\right) = -\frac{1}{u^2}\int_0^u \left(f^*(\tau) - f^*(u)\right)d\tau$$

a.e. in (0, 8). Therefore, a change of variables and Hölder's inequality show that the last term in (33) can be estimated from above (up to multiplicative positive constants) by

$$|h|\left(\int_{V_n 2^n|h|^n}^{\infty} s^{\left(1-\frac{1}{n}\right)(1+\epsilon)} s^{-2(1+\epsilon)}\left(\int_0^{3^n s/2^n} \left(f^*(\tau) - f^*(3^n s/2^n)\right)d\tau\right)^{1+\epsilon} ds\right)^{\frac{1}{1+\epsilon}}$$

$$\approx |h|\left(\int_{V_n 3^n|h|^n}^{\infty} u^{-(1+\epsilon)-\frac{1+\epsilon}{n}}\left(\int_0^u \left(f^*(\tau) - f^*(u)\right)d\tau\right)^{1+\epsilon} du\right)^{\frac{1}{1+\epsilon}}$$

$$\le |h|\left(\int_{V_n 3^n|h|^n}^{\infty} u^{-1-\frac{1+\epsilon}{n}}\int_0^u \left(f^*(\tau) - f^*(u)\right)^{1+\epsilon} d\tau\, du\right)^{\frac{1}{1+\epsilon}}. \tag{34}$$

$$+(1+\epsilon)\left(\int_{V_n 3^n(1+\epsilon)^n}^{\infty} s^{1-\frac{1+\epsilon}{n}} \int_0^s \left(f^*(u)-f^*(s)\right)^{1+\epsilon} du\, ds\right)^{\frac{1}{1+\epsilon}}. \quad (35)$$

We claim that the latter sum is dominated by its last term. Indeed, we obtain by means of Remark 4.4 (recall that $0 < \epsilon < \infty$) and Proposition 4.2 that, for all $\epsilon \geq 0$,

$$\left(\int_0^{V_n 3^n(1+\epsilon)^n} \left(f^{**}(u)-f^{**}(V_n 3^n(1+\epsilon)^n)\right)^{1+\epsilon} du\right)^{\frac{1}{1+\epsilon}} \leq \left(\int_0^{V_n 3^n(1+\epsilon)^n} \left(f^{**}(u)-f^{**}(V_n 3^n(1+\epsilon)^n)\right)^{1+\epsilon} du\right)^{\frac{1}{1+\epsilon}}$$

$$= \left(\int_0^{V_n 3^n(1+\epsilon)^n} \left(\frac{1}{u}\int_0^u f^*(s)-f^*(V_n 3^n(1+\epsilon)^n) ds\right)^{1+\epsilon} du\right)^{\frac{1}{1+\epsilon}}$$

$$\lesssim \left(\int_0^{V_n 3^n(1+\epsilon)^n} \left(f^*(u)-f^*(V_n 3^n(1+\epsilon)^n)\right)^{1+\epsilon} du\right)^{\frac{1}{1+\epsilon}}$$

$$\approx \left(\int_{V_n 3^n(1+\epsilon)^n}^{\infty} s^{-1-\frac{1+\epsilon}{n}} ds\right)^{\frac{1}{1+\epsilon}} \left(\int_0^{V_n 3^n(1+\epsilon)^n} \left(f^*(u)-f^*(V_n 3^n(1+\epsilon)^n)\right)^{1+\epsilon} du\right)^{\frac{1}{1+\epsilon}}$$

$$= (1+\epsilon)\left(\int_{V_n 3^n(1+\epsilon)^n}^{\infty} s^{-1-\frac{1+\epsilon}{n}} \int_0^{V_n 3^n(1+\epsilon)^n} \left(f^*(u)-f^*(V_n 3^n(1+\epsilon)^n)\right)^{1+\epsilon} du\, ds\right)^{\frac{1}{1+\epsilon}}$$

$$\leq (1+\epsilon)\left(\int_{V_n 3^n(1+\epsilon)^n}^{\infty} s^{-1-\frac{1+\epsilon}{n}} \int_0^{s} \left(f^*(u)-f^*(s)\right)^{1+\epsilon} du\, ds\right)^{\frac{1}{1+\epsilon}}.$$

To complete our proof, note that the factor $V_n 3^n$ can be omitted from the last term in (35) (as follows by arguments used in the discussion following (32)).

6. Proof of Proposition 3.6

We shall start with the following result [17]:

Lemma 6.1. Let $0 \leq \epsilon < \infty$, and $\frac{\epsilon+\epsilon^2-1}{1+\epsilon} \in \mathbb{R}$. Let $f \in \mathcal{M}_0(\mathbb{R}^n)$ satisfy $|\mathrm{supp}\, f|_n \leq 1$ and

$$\left\|(1+\epsilon)^{\frac{\epsilon}{1+\epsilon}} \ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}(1+\epsilon)\left(\int_{(1+\epsilon)^n}^{2} \& s^{-\frac{1+\epsilon}{n}} \int_0^s \& \left(f^* \& (u)-f^*(s)\right)^{1+\epsilon} du \& \frac{ds}{s}\right)^{\frac{1}{1+\epsilon}}\right\|_{1+\epsilon,(0,1)}. \quad (36)$$

Then $f \in L_{1+\epsilon}$ and the function F defined by

$$F(x^2) = f^*(V_n|x^2|^n) \text{ if } \epsilon = 0 \quad \text{or} \quad F(x^2) = f^{**}(V_n|x^2|^n) \text{ if } 0 < \epsilon < \infty \quad (37)$$

belongs to $B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon}$. Moreover,

$$\|F\|_{B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon}} \lesssim \left\|(1+\epsilon)^{\frac{\epsilon}{1+\epsilon}} \ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}(1+\epsilon) \times \left(\int_{(1+\epsilon)^n}^{2} s^{-\frac{1+\epsilon}{n}} \int_0^s \left(f^*(u)-f^*(s)\right)^{1+\epsilon} du \frac{ds}{s}\right)^{\frac{1}{1+\epsilon}}\right\|_{1+\epsilon,(0,1)} \quad (38)$$

for all f mentioned above.

Proof. Take $f \in \mathcal{M}_0(\mathbb{R}^n)$with $|\mathrm{supp} f|_n \leq 1$. Then $f^*(1+\epsilon) = 0$ for $\epsilon \geq 0$. Therefore, when $s \in (1,\infty)$, $\int_0^s \left(f^*(u)-f^*(s)\right)^{1+\epsilon} du = \int_0^s f^*(u)^{1+\epsilon} du = \int_0^1 f^*(u)^{1+\epsilon} du$. Hence,

$$\|f\|_{1+\epsilon} = \left(\int_0^1 f^*(u)^{1+\epsilon} du\right)^{\frac{1}{1+\epsilon}} \approx \left(\int_1^2 s^{-\frac{1+\epsilon}{n-1}} ds \int_0^1 f^*(u)^{1+\epsilon} du\right)^{\frac{1}{1+\epsilon}} \approx$$

$$\left\|(1+\epsilon)^{\frac{\epsilon}{1+\epsilon}}\ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}(1+\epsilon)\right\|_{1+\epsilon,(0,1)}\left(\int_1^2 s^{-\frac{1+\epsilon}{n-1}}\int_0^s \left(f^*(u)-f^*(s)\right)^{1+\epsilon} du\, ds\right)^{\frac{1}{1+\epsilon}}$$
$$\leq \left\|(1+\epsilon)^{\frac{\epsilon}{1+\epsilon}}\ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}(1+\epsilon)\left(\int_{(1+\epsilon)^n}^2 s^{-\frac{1+\epsilon}{n-1}}\int_0^s \left(f^*(u)-f^*(s)\right)^{1+\epsilon} du\, ds\right)^{\frac{1}{1+\epsilon}}\right\|_{1+\epsilon,(0,1)}. \quad (39)$$

Together with (36), this shows that $f \in L^{1+\epsilon}$.
On the other hand, using (39),

$$\left\|(1+\epsilon)^{\frac{\epsilon}{1+\epsilon}}\ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}(1+\epsilon)\left(\int_{(1+\epsilon)^n}^\infty s^{-\frac{1+\epsilon}{n}}\int_0^s \left(f^*(u)-f^*(s)\right)^{1+\epsilon} du\frac{ds}{s}\right)^{\frac{1}{1+\epsilon}}\right\|_{1+\epsilon,(0,1)}$$
$$\leq \left\|(1+\epsilon)^{\frac{\epsilon}{1+\epsilon}}\ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}(1+\epsilon)\left(\int_{(1+\epsilon)^n}^1 s^{-\frac{1+\epsilon}{n}}\int_0^s \left(f^*(u)-f^*(s)\right)^{1+\epsilon} du\frac{ds}{s}\right)^{\frac{1}{1+\epsilon}}\right\|_{1+\epsilon,(0,1)}$$
$$+\left\|(1+\epsilon)^{\frac{\epsilon}{1+\epsilon}}\ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}(1+\epsilon)\left(\int_{(1+\epsilon)^n}^\infty s^{-\frac{1+\epsilon}{n}}\int_0^1 f^*(u)^{1+\epsilon} du\frac{ds}{s}\right)^{\frac{1}{1+\epsilon}}\right\|_{1+\epsilon,(0,1)}$$
$$\lesssim \left\|(1+\epsilon)^{\frac{\epsilon}{1+\epsilon}}\ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}(1+\epsilon)\left(\int_{(1+\epsilon)^n}^2 s^{-\frac{1+\epsilon}{n}}\int_0^s \left(f^*(u)-f^*(s)\right)^{1+\epsilon} du\frac{ds}{s}\right)^{\frac{1}{1+\epsilon}}\right\|_{1+\epsilon,(0,1)} \quad (40)$$

Now, since $\|F\|_{1+\epsilon} \lesssim \|f\|_{1+\epsilon}$, (38) follows from Proposition 3.5 and estimates (39) and (40).

Proof of Proposition 3.6 (see [17]).

Step 1: Assume that (9) holds. Take $f \in \mathcal{M}_0(\mathbb{R}^n)$with $|\mathrm{supp} f|_n \leq 1$. Then either the right-hand side of (10) is finite or infinite. If it is infinite, (10) is clear. So assume that the right-hand side of (10) is finite. In such a case, we apply Lemma 6.1 to get that the function F given by (37) satisfies (38). Using hypothesis (9) with F instead of f and the estimate $F^*(1+\epsilon) \geq f^*(1+\epsilon)$, inequality (10) follows.

Step 2: Assume now that (10) holds. Take $f \in B_{1+\epsilon,1+\epsilon}^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}$with $|\mathrm{supp}\, f|_n \leq 1$. Since $f \in L_{1+\epsilon}$, Proposition 4.7 and (10) yield (9).

Consider now a general $f \in B_{1+\epsilon,1+\epsilon}^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}$ and put $g(x^2) := f^*(V_n|x^2|^n)_{\chi[0,1)}(V_n|x^2|^n), x^2 \in \mathbb{R}^n$. Clearly, $|\mathrm{supp}\, g|_n \leq 1$and $g^*(1+\epsilon) = f^*(1+\epsilon)_{\chi[0,1)}(1+\epsilon), \epsilon \geq -1$. In particular, $g \in L_{1+\epsilon}$. Applying our hypothesis (10) to g instead of f and using Proposition 4.7, we arrive at

$$\left\|\omega^2(1+\epsilon)f^*(1+\epsilon)_{\chi[0,1)}(1+\epsilon)\right\|_{1+2\epsilon,(0,1)}\left\|(1+\epsilon)^{\frac{\epsilon}{1+\epsilon}}\ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}(1+\epsilon)\left(\int_{(1+\epsilon)^n}^1 s^{-\frac{1+\epsilon}{n}}\int_0^s \left(f^*(u)-f^*(s)\right)^{1+\epsilon} du\frac{ds}{s}\right)^{\frac{1}{1+\epsilon}}\right\|_{1+\epsilon,(0,1)}$$
$$+\left\|(1+\epsilon)^{\frac{\epsilon}{1+\epsilon}}\ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}(1+\epsilon)\left(\int_{(1+\epsilon)^n}^2 s^{-\frac{1+\epsilon}{n}}\int_0^1 f^*(u)^{1+\epsilon} du\frac{ds}{s}\right)^{\frac{1}{1+\epsilon}}\right\|_{1+\epsilon,(0,1)}$$
$$\lesssim \left\|(1+\epsilon)^{-\frac{1}{1+\epsilon}}\ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}(1+\epsilon)\omega_1^2(f,1+\epsilon)_{1+\epsilon}\right\|_{1+\epsilon,(0,1)} + \|f\|_{1+\epsilon} = \|f\|_{B_{1+\epsilon,1+\epsilon}^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}} \quad (41)$$

and (9) follows.

7. Proof of Theorem 3.1

To prove Theorem 3.1 (see [17]), we shall need a variant of Lemma 6.1. This is why we start with the following:

Remark 7.1. Lemma 6.1 continues to hold if we assume additionally that $\epsilon \geq 0$ and if expression (36) is replaced by

$$+\left\|(1+\epsilon)^{\frac{2\epsilon}{1+2\epsilon}}\ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}(1+\epsilon)\left(\int_{(1+\epsilon)^n}^{2} s^{-\frac{1+\epsilon}{n}}\int_{(1+\epsilon)^n}^{s} f^*(u)^{1+\epsilon}du\frac{ds}{s}\right)^{\frac{1}{1+\epsilon}}\right\|_{1+2\epsilon,(0,1)}$$

$$\lesssim\left\|(1+\epsilon)^{\frac{2\epsilon}{1+2\epsilon}}\ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}(1+\epsilon)\left(\int_0^{(1+\epsilon)^n} f^*(u)^{1+\epsilon}du\right)^{\frac{1}{1+\epsilon}}\right\|_{1+2\epsilon,(0,1)}$$

$$+\left\|(1+\epsilon)^{\frac{2\epsilon}{1+2\epsilon}}\ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}(1+\epsilon)\left(\int_{(1+\epsilon)^n}^{2} f^*(u)^{1+\epsilon}\int_u^2 s^{-\frac{1+\epsilon}{n-1}}ds\,du\right)^{\frac{1}{1+\epsilon}}\right\|_{1+2\epsilon,(0,1)}$$

$$\approx\left\|(1+\epsilon)^{\frac{2\epsilon}{1+2\epsilon}}\ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}(1+\epsilon)\left(\int_0^{1+\epsilon} f^*(u)^{1+\epsilon}du\right)^{\frac{1}{1+\epsilon}}\right\|_{1+2\epsilon,(0,1)}$$

$$+\left\|(1+\epsilon)^{\frac{1+\epsilon}{n}-\frac{1+\epsilon}{1+2\epsilon}}\ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}(1+\epsilon)\int_{1+\epsilon}^{1} u^{-\frac{1+\epsilon}{n}}f^*(u)^{1+\epsilon}\,du\right\|_{1+2\epsilon,(0,1)}^{\frac{1}{1+\epsilon}}$$

$$+\left\|(1+\epsilon)^{\frac{\epsilon}{1+2\epsilon}}\ell^{\epsilon+\epsilon^2-1}(1+\epsilon)f^*(1+\epsilon)^{1+\epsilon}\right\|_{\frac{1+2\epsilon}{1+\epsilon},(0,1)}^{\frac{1}{1+\epsilon}}$$

$$\lesssim\left\|(1+\epsilon)^{-\frac{1}{1+2\epsilon}}\ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}(1+\epsilon)\left(\int_0^{1+\epsilon} f^*(u)^{1+\epsilon}\,du\right)^{\frac{1}{1+\epsilon}}\right\|_{1+2\epsilon,(0,1)}$$

So, the conclusion follows immediately from Lemma 6.1.

Proof of Theorem 3.1.

Step 1: Here we prove the sufficiency of the condition $\epsilon\geq 0$ under the additional assumption (see [17]).

Due to Proposition 4.7, it is enough to show that, for all $f\in B_{1+\epsilon,1+\epsilon}^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}$,

$$\left\|(1+\epsilon)^{\frac{\epsilon}{(1+\epsilon)(1+2\epsilon)}}\ell^{\frac{\epsilon}{1+\epsilon}}(1+\epsilon)f^*(1+\epsilon)\right\|_{1+2\epsilon,(0,1)}$$
$$\lesssim\|f\|_{1+\epsilon}$$
$$+\left\|(1+\epsilon)^{-\frac{1}{1+\epsilon}}\ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}(1+\epsilon)\right.$$
$$\left.\times\left(\int_{(1+\epsilon)^n}^{\infty} s^{-\frac{1+\epsilon}{n}}\int_0^s\left(f^*(u)-f^*(s)\right)^{1+\epsilon}du\frac{ds}{s}\right)^{\frac{1}{1+\epsilon}}\right\|_{1+\epsilon,(0,1)}\tag{42}$$

(i) First consider the case $\epsilon=0$.

Since $\int_0^{(1-\epsilon)^n} f^*(u)du\lesssim(1-\epsilon)\int_{(1-\epsilon)^n}^{\infty} s^{-1/n}\int_0^s\left(f^*(u)-f^*(s)\right)du\frac{ds}{s})$ (cf. (32) with $V_n^{\frac{1}{n}}|h|$ replaced by $0<\epsilon<1$), we see that it is sufficient to prove that, for all $f\in B_{1+\epsilon,1+\epsilon}^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}$,

$$\left\|(1-\epsilon)^{\frac{2\epsilon}{1+2\epsilon}}\ell^{\epsilon}(1-\epsilon)f^*(1-\epsilon)\right\|_{1+2\epsilon,(0,1)}$$
$$\lesssim\|f\|_1+\left\|(1-\epsilon)^{-\frac{1}{1+\epsilon}}\ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}(1-\epsilon)\int_0^{(1-\epsilon)^n} f^*(s)\,ds\right\|_{1+\epsilon,(0,1)}.\tag{43}$$

If $\epsilon=\infty$, then, by our assumption, also $\epsilon=\infty$ and (43) is trivial. Thus, we suppose that $0\leq\epsilon<\infty$. For simplicity, we consider only the case when $\epsilon<\infty$ (the case $\epsilon=\infty$ can be handled similarly). Using the fact that

$$\lesssim \|f\|_1 + \left(\sum_{k=0}^{\infty} \ell^{\epsilon(1+2\epsilon)} \left(2^{-2^{k+1}}\right) \sum_{i=0}^{2^k-1} \left(2^{-2^{k+1}} 2^i\right)^{1+2\epsilon} f^* \left(2^{-2^{k+1}} 2^i\right)^{1+2\epsilon} \right)^{\frac{1}{1+2\epsilon}}$$

Write $\frac{1}{1+2\epsilon} = \left(\frac{1+\epsilon}{1+2\epsilon}\right)\left(\frac{1}{1+\epsilon}\right)$, take the exponent $\frac{1+\epsilon}{1+2\epsilon}$ inside the outer sum and afterwards take the factor $\frac{1}{1+2\epsilon}$ of this exponent inside the inner sum (all this is possible because we are assuming $\epsilon \geq 0$), to get

$$\left\| (1-\epsilon)^{\frac{2\epsilon}{1+2\epsilon}} \ell^{\epsilon}(1-\epsilon) f^*(1-\epsilon) \right\|_{1+2\epsilon,(0,1)}$$

$$\lesssim \|f\|_1 + \left(\sum_{k=0}^{\infty} \ell^{\epsilon+\epsilon^2} \left(2^{-2^k}\right) \left(\sum_{i=0}^{2^k-1} 2^{-2^{k+1}} 2^i f^* \left(2^{-2^{k+1}} 2^i\right) \right)^{1+\epsilon} \right)^{\frac{1}{1+\epsilon}}$$

$$\lesssim \|f\|_1 + \left(\sum_{k=0}^{\infty} \ell^{\epsilon+\epsilon^2} \left(2^{-2^k}\right) \left(\sum_{i=0}^{2^k-1} \int_{2^{-2^{k+1}} 2^{i-1}}^{2^{-2^k} 2^i} f^* \left(2^{-2^{k+1}} 2^i\right) \right)^{1+\epsilon} \right)^{\frac{1}{1+\epsilon}}$$

$$\leq \|f\|_1 + \left(\sum_{k=0}^{\infty} \ell^{\epsilon+\epsilon^2} \left(2^{-2^k}\right) \left(\int_0^{2^{-2^k}} f^*(1-\epsilon) \right)^{1+\epsilon} \right)^{\frac{1}{1+\epsilon}}$$

$$\approx \|f\|_1 + \left(\sum_{k=0}^{\infty} \ell^{\epsilon+\epsilon^2} \left(2^{-2^{k+1}}\right) - \ell^{\epsilon+\epsilon^2} \left(2^{-2^k}\right) \left(\int_0^{2^{-2^k}} f^*(s) \right)^{1+\epsilon} \right)^{\frac{1}{1+\epsilon}}$$

$$\lesssim \|f\|_1 + \left(\sum_{k=0}^{\infty} \int_{2^{-2^{k+1}}}^{2^{-2^k}} \ell^{\epsilon+\epsilon^2-1}(1-\epsilon) \left(\int_0^{1-\epsilon} f^*(s) \right)^{1+\epsilon} \frac{d(1+\epsilon)}{1-\epsilon} \right)^{\frac{1}{1+\epsilon}}$$

$$\leq \|f\|_1 + \left\| (1-\epsilon)^{-\frac{1}{1+\epsilon}} \ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}} (1+\epsilon) \int_0^{1-\epsilon} f^{*(s)ds} \right\|_{1+\epsilon,(0,1)}$$

which, after a change of variables, proves (43).

(ii) Now consider the case $0 < \epsilon < \infty$. Using the monotonicity of function (13), we obtain, for all $\epsilon \geq 0$,

$$\left(\int_0^{(1+\epsilon)^n} \left(f^*(s) - f^*((1+\epsilon)^n)\right)^{1+\epsilon} ds \right)^{\frac{1}{1+\epsilon}}$$

$$\lesssim (1+\epsilon) \left(\int_{(1+\epsilon)^n}^{\infty} s^{-\frac{1+\epsilon}{n}} \int_0^s \left(f^*(u) - f^*(s)\right)^{1+\epsilon} du \frac{ds}{s} \right)^{\frac{1}{1+\epsilon}}$$

Thus, in view of (42), it is enough to prove that

$$\left\|(1+\epsilon)^{\frac{2\epsilon}{1+2\epsilon}}\ell^{\epsilon}(1+\epsilon)f^*(1+\epsilon)\right\|_{1+2\epsilon,(0,1)}$$
$$\lesssim \left\|(1+\epsilon)^{-\frac{1}{1+\epsilon}}\ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}(1+\epsilon)\left(\int_0^{(1+\epsilon)^n}\left(f^*(s)-f^*(1+\epsilon)^n)\right)^{1+\epsilon}ds\right)^{\frac{1}{1+\epsilon}}\right\|_{1+\epsilon,(0,1)} \tag{44}$$

Applying the estimate $f^* \le f^{**}$, (16) and the Hardy-type inequality from Proposition 4.6(ii), we arrive at

$$\left\|(1+\epsilon)^{\frac{\epsilon}{(1+\epsilon)(1+2\epsilon)}}\ell^{\frac{\epsilon}{1+\epsilon}}(1+\epsilon)f^*(1+\epsilon)\right\|_{1+2\epsilon,(0,1)}$$
$$\le \left\|(1+\epsilon)^{\frac{\epsilon}{(1+\epsilon)(1+2\epsilon)}}\ell^{\frac{\epsilon}{1+\epsilon}}(1+\epsilon)\left(f^{**}(1)+\left(f^{**}(1+\epsilon)-f^{**}(1)\right)\right)\right\|_{1+2\epsilon,(0,1)}$$
$$\lesssim \|f\|_{1+\epsilon}+\left\|(1+\epsilon)^{\frac{\epsilon}{(1+\epsilon)(1+2\epsilon)}}\ell^{\frac{\epsilon}{1+\epsilon}}(1+\epsilon)o\left(\left(f^{**}(1+\epsilon)-f^{**}(1)\right)\right)\right\|_{1+2\epsilon,(0,1)}$$
$$= \|f\|_{1+\epsilon}+\left\|(1+\epsilon)^{\frac{\epsilon}{(1+\epsilon)(1+2\epsilon)}}\ell^{\frac{\epsilon}{1+\epsilon}}(1+\epsilon)\left(\int_{1+\epsilon}^1\frac{f^{**}(s)\ \ f^*(s)}{s}ds\right)\right\|_{1+2\epsilon,(0,1)}$$
$$\lesssim \|f\|_{1+\epsilon}+\left\|(1+\epsilon)^{\frac{\epsilon}{(1+\epsilon)(1+2\epsilon)}}\ell^{\epsilon}(1+\epsilon)\left(f^{**}(1+\epsilon)\ \ -f^*(1+\epsilon)\right)\right\|_{1+2\epsilon,(0,1)} \tag{45}$$

If $\epsilon = \infty$, we use Hölder's inequality to get

$$f^{**}(1+\epsilon)-f^*(1+\epsilon)\le(1+\epsilon)^{-\frac{1}{1+\epsilon}}\left(\int_0^{1+\epsilon}\left(f^*(u)-f^*(1+\epsilon)\right)^{1+\epsilon}\right)^{\frac{1}{1+\epsilon}}.$$

Consequently

$$\left\|(1+\epsilon)^{\frac{2\epsilon}{(1+\epsilon)(1+2\epsilon)}}\ell^{\frac{\epsilon}{1+\epsilon}}(1+\epsilon)f^*(1+\epsilon)\right\|_{1+2\epsilon,(0,1)}$$
$$\lesssim \|f\|_{1+\epsilon}+\left\|(1+\epsilon)^{\frac{2\epsilon}{(1+\epsilon)(1+2\epsilon)}}\ell^{\epsilon}(1+\epsilon)\left(\int_0^{1+\epsilon}\left(f^*(u)-f^*(1+\epsilon)\right)^{1+\epsilon}du\right)^{\frac{1}{1+\epsilon}}\right\|_{1+2\epsilon,(0,1)},$$

and (44) follows immediately since our assumption $\epsilon \ge 0$ implies that also $\epsilon = \infty$.

If $0 \le \epsilon < \infty$, then (45), the obvious estimate

$$\left\|(1+\epsilon)^{\frac{2\epsilon}{(1+\epsilon)(1+2\epsilon)}}\ell^{\epsilon}(1+\epsilon)\left(f^{**}(1+\epsilon)-f^*(1+\epsilon)\right)\right\|_{1+2\epsilon,\left(\frac{1}{4},1\right)}\le\|f\|_{1+\epsilon}.$$

Eq. (44) and Proposition 4.5 show that it is enough to prove that

$$\left\|(1+\epsilon)^{\frac{\epsilon}{(1+\epsilon)(1+2\epsilon)}}\ell^{\epsilon}(1+\epsilon)\left(f^{**}(1+\epsilon)-f^*(1+\epsilon)\right)\right\|_{1+2\epsilon,\left(0,\frac{1}{4}\right)}$$
$$\lesssim \left\|(1+\epsilon)^{-\frac{1}{1+\epsilon}}\ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}(1+\epsilon)\left(\int_0^{T^n}\left(f^{**}(s)-f^*(s)\right)^{1+\epsilon}ds\right)\right\|_{1+\epsilon,(0,1)} \tag{46}$$

For simplicity, we consider only the case when $\epsilon < \infty$ (the case $\epsilon = \infty$ can be handled similarly). Having the monotonicity of function (14) in mind, we obtain

$$\left\|(1+\epsilon)^{\frac{\epsilon}{(1+\epsilon)(1+2\epsilon)}}\ell^{\epsilon}(1+\epsilon)\left(f^{**}(1+\epsilon)-f^*(1+\epsilon)\right)\right\|_{1+2\epsilon,\left(0,\frac{1}{4}\right)}$$
$$=\left(\sum_{k=1}^{\infty}\int_{2^{-2^{k+1}}}^{2^{-2^k}}(1+\epsilon)^{\frac{\epsilon}{1+\epsilon}}\ell^{\epsilon(1+2\epsilon)}(1+\epsilon)\left(f^{**}(1+\epsilon)-f^*(1+\epsilon)\right)^{1+2\epsilon}d(1+\epsilon)\right)^{\frac{1}{1+2\epsilon}}$$

$$+\epsilon)\Bigg)^{\frac{1}{1+2\epsilon}}$$

$$\lesssim\left(\sum_{k=1}^{\infty}\ell^{\epsilon(1+2\epsilon)}\left(2^{-2^{k}}\right)\sum_{i=0}^{2^{k}-1}\left(2^{-2^{k+1}}2^{i}\right)^{\frac{1+2\epsilon}{1+\epsilon}}\left(f^{**}\left(2^{-2^{k+1}}2^{i+1}\right)\right.\right.$$

$$\left.\left.-f^{*}\left(2^{-2^{k+1}}2^{i+1}\right)\right)^{1+2\epsilon}\right)^{\frac{1}{1+2\epsilon}}$$

(47)

Write $\frac{1}{1+2\epsilon}=\left(\frac{1+\epsilon}{1+2\epsilon}\right)\left(\frac{1}{1+\epsilon}\right)$ and take the exponent $\frac{1+\epsilon}{1+2\epsilon}$ inside the outer sum (since $\epsilon\geq 0$). Then the inner sum will have the exponent $\frac{1+\epsilon}{1+2\epsilon}$, which we write as $\left(\frac{1+\epsilon}{1+2\epsilon}\right)\left(\frac{1+\epsilon}{1+\epsilon}\right)$ and then take its factor $\frac{1+\epsilon}{1+2\epsilon}$ inside the inner sum (since $\epsilon\geq 0$). This leads to an upper estimate by

$$\left(\sum_{k=1}^{\infty}\ell^{\epsilon+\epsilon^{2}}\left(2^{-2^{k}}\right)\left(\sum_{i=0}^{2^{k}-1}\left(2^{-2^{k+1}}2^{i}\right)\left(f^{**}\left(2^{-2^{k+1}}2^{i+1}\right)-f^{*}\left(2^{-2^{k+1}}2^{i+1}\right)\right)^{1+\epsilon}\right)\right)^{\frac{1}{1+\epsilon}}$$

The estimate $a=a^{1+\epsilon}a^{-\epsilon}\approx a^{1+\epsilon}\int_{2a}^{4a}(1+\epsilon)^{-(1+\epsilon)}d(1+\epsilon)$, for all $a:=2^{-2^{k+1}}2^{i}$, and the monotonicity of functions (14) and (13) allow to dominate the last expression (up to a multiplicative positive constant) by

$$\left(\sum_{k=1}^{\infty}\ell^{\epsilon+\epsilon^{2}}\left(2^{-2^{k}}\right)\left(\sum_{i=0}^{2^{k}-1}\int_{2^{-2^{k+1}}2^{i+1}}^{2^{-2^{k+1}}2^{i+2}}\left(f^{**}(1+\epsilon)-f^{*}(1+\epsilon)\right)^{1+\epsilon}d(1+\epsilon)\right)\right)^{\frac{1}{1+\epsilon}}$$

$$\leq\left(\sum_{k=1}^{\infty}\ell^{\epsilon+\epsilon^{2}}\left(2^{-2^{k+1}}\right)\left(\int_{0}^{2^{-2^{k+1}}+1}\left(f^{**}(s)-f^{*}(s)\right)^{1+\epsilon}ds\right)\right)^{\frac{1}{1+\epsilon}}$$

$$\approx\left(\sum_{k=1}^{\infty}\ell^{\epsilon+\epsilon^{2}}\left(2^{-2^{k+1}}+1\right)-\ell^{\epsilon+\epsilon^{2}}\left(2^{-2^{k}}+1\right)\left(\int_{0}^{2^{-2^{k+1}}+1}\left(f^{**}(s)-f^{*}(s)\right)^{1+\epsilon}ds\right)\right)^{\frac{1}{1+\epsilon}}$$

$$\lesssim\left(\sum_{k=1}^{\infty}\int_{2^{-2^{k+1}}+1}^{2^{-2^{k}}+1}\ell^{\epsilon+\epsilon^{2}}((1+\epsilon))(1+\epsilon)^{-1}\left(\int_{0}^{1+\epsilon}\left(f^{**}(s)-f^{*}(s)\right)^{1+\epsilon}ds\right)\right)^{\frac{1}{1+\epsilon}}$$

$$=\left\|(1+\epsilon)^{-\frac{1}{1+\epsilon}}\ell^{\frac{\epsilon+\epsilon^{2}-1}{1+\epsilon}}(1+\epsilon)\left(\int_{0}^{1+\epsilon}\left(f^{**}(s)-f^{*}(s)\right)^{1+\epsilon}ds\right)^{\frac{1}{1+\epsilon}}\right\|_{1+\epsilon,(0,1)}$$

which, after a change of variables, together with the estimates obtained above, gives (46).

Step 2: Now, we prove the sufficiency of the condition $\epsilon\geq 0$ even when $\epsilon>0$. Thus, assume that $\epsilon>0$. It is enough to prove (42) (for all $\epsilon>0$) but with

$$\left\|(1+\epsilon)^{-\frac{\epsilon}{(1+3\epsilon)(1+2\epsilon)}}\ell^{\frac{\epsilon^{2}(5+6\epsilon)}{(1+3\epsilon)(1+2\epsilon)}}(1+\epsilon)f^{*}(1+\epsilon)\right\|_{1+2\epsilon,(0,1)}$$

on its left-hand side.

Essentially, we can follow part (ii) of Step 1. The only modifications are that the case $\epsilon=\infty$ does not occur and also the way used to estimate the expression corresponding to the last term in (47) by (48) is a different then

$$f_{y^2}^*(1+\epsilon) = y^{-\frac{2}{1+3\epsilon}}\ell^{-\frac{\epsilon+3e^2-1}{1+3\epsilon}}(y^2)\chi_{[0,y^2]}(1+\epsilon) + (1+\epsilon)^{-\frac{1}{1+3\epsilon}}\ell^{-\frac{\epsilon+3e^2-1}{1+3\epsilon}}(1+\epsilon)\chi_{(y^2,\chi)}(1+\epsilon), \qquad \epsilon \geq 0.$$

(i) Case $0 < \epsilon < \infty$.

Defining$F_{y^2}(x^2) = f_{y^2}^{**}(V_n|x^2|^n)$, $x^2 \in \mathbb{R}^n$, we get

$$\|F_{y^2}\|_{1+\epsilon} = \left\|F_{y^2}^*\right\|_{1+\epsilon} = \left\|f_{y^2}^{**}\right\|_{1+\epsilon} \lesssim \left\|f_{y^2}^*\right\|_{1+\epsilon} \lesssim \ell^{-\epsilon}(\omega^2) \approx 1 \text{ for all} y^2 \in \left(0,\frac{\omega^2}{2}\right).$$

Moreover, Proposition 3.5(ii), a change of variables, the triangle inequality and the fact that $f_{y^2}^*$ is constant in $(0,y^2)$ imply that

$$\begin{aligned}
&\left\|(1+\epsilon)^{-\frac{1}{1+\epsilon}}\ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}(1+\epsilon)\omega_1^2(F_{y^2},1+\epsilon)_{1+\epsilon}\right\|_{1+\epsilon,(0,1)} \\
&\quad \lesssim \Bigg\|(1+\epsilon)^{\frac{\epsilon}{1+\epsilon}}\ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}((1+\epsilon)) \\
&\quad \times (1+\epsilon)^{\frac{1}{n}}\left(\int_{1+\epsilon}^{\infty} s^{-\frac{1+\epsilon}{n-1}}\int_0^s \left(f_{y^2}^*(u)-f_{y^2}^*(s)\right)^{1+\epsilon} du\, ds\right)^{\frac{1}{1+\epsilon}}\Bigg\|_{1+\epsilon,(0,1)} \\
&\lesssim \left\|(1+\epsilon)^{\frac{\epsilon}{1+\epsilon}}\ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}((1+\epsilon))(1+\epsilon)^{\frac{1}{n}}\right\|_{1+\epsilon,(0,y^2)}\left(\int_{1+\epsilon}^{\infty} s^{-\frac{1+\epsilon}{n-1}}\int_0^s \left(f_{y^2}^*(u)-f_{y^2}^*(s)\right)^{1+\epsilon} du\, ds\right)^{\frac{1}{1+\epsilon}} \\
&\quad + \Bigg\|(1+\epsilon)^{\frac{\epsilon}{1+\epsilon}}\ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}((1+\epsilon)) \\
&\quad \times (1+\epsilon)^{\frac{1}{n}}\left(\int_{1+\epsilon}^{\infty} s^{-\frac{1+\epsilon}{n-1}}\int_0^s \left(f_{y^2}^*(u)-f_{y^2}^*(s)\right)^{1+\epsilon} du\, ds\right)^{\frac{1}{1+\epsilon}}\Bigg\|_{1+\epsilon,(y^2,1)} = A+B.
\end{aligned} \tag{49}$$

Furthermore, since $f_{y^2}^*(u) - f_{y^2}^*(s) \leq f_{y^2}^*(u)$,

$$\begin{aligned}
A &\lesssim y^{\frac{2}{n}}\ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}(y^2)\left(\int_{y^2}^{\infty} s^{--\frac{1+\epsilon}{n-1}}\int_{y^2}^{s} f_{y^2}^*(u)^{1+\epsilon}\, du\, ds + \int_{y^2}^{\infty} s^{-\frac{1+\epsilon}{n-1}}\int_0^{y^2} f_{y^2}^*(u)^{1+\epsilon}\, du\, ds\right)^{\frac{1}{1+\epsilon}} \\
&\lesssim y^{\frac{2}{n}}\ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}(y^2)\left(y^{-\frac{2(1+\epsilon)}{n}}\ell^{-(\epsilon+\epsilon^2)}(y^2) + y^{-\frac{2(1+\epsilon)}{n}}\ell^{-(\epsilon+\epsilon^2+1)}(y^2)\right)^{\frac{1}{1+\epsilon}} \lesssim \ell^{-\frac{1}{1+\epsilon}}(y^2) \lesssim 1(50)
\end{aligned}$$

and

$$\begin{aligned}
B &\lesssim \left\|(1+\epsilon)^{\frac{\epsilon}{1+\epsilon}}\ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}(1+\epsilon)(1+\epsilon)^{\frac{1}{n}}\left(\int_{1+\epsilon}^{\infty} s^{-\frac{1+\epsilon}{n-1}}\int_{y^2}^{s} f_{y^2}^*(u)^{1+\epsilon}du\, ds\right)^{\frac{1}{1+\epsilon}}\right\|_{1+\epsilon,(y^2,1)} \\
&\quad + \left\|(1+\epsilon)^{\frac{\epsilon}{1+\epsilon}}\ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}(1+\epsilon)(1+\epsilon)^{\frac{1}{n}}\left(\int_{1+\epsilon}^{\infty} s^{-1+\epsilon/n-1}\int_0^{s} f_{y^2}^*(u)^{1+\epsilon}du\, ds\right)^{\frac{1}{1+\epsilon}}\right\|_{1+\epsilon,(y^2,1)} \\
&\lesssim \left\|(1+\epsilon)^{-\frac{1}{1+\epsilon}}\ell^{-\frac{1}{1+\epsilon}}(1+\epsilon)\right\|_{1+\epsilon,(y^2,1)} + \left\|(1+\epsilon)^{-\frac{1}{1+\epsilon}}\ell^{\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}(1+\epsilon)\right\|_{1+\epsilon,(y^2,1)}\ell^{-\frac{\epsilon+\epsilon^2+1}{1+\epsilon}}(y^2) \\
&\lesssim (ln\,(y^2))^{\frac{1}{1+\epsilon}} \qquad (51)
\end{aligned}$$

: $\left(0,\frac{\omega^2}{2}\right)$. Therefore $F_{y^2} \in B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon}$ and $\|F_{y^2}\|_{B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon}} \lesssim (\ln(y^2))^{\frac{1}{1+\epsilon}}$ for all

, the inequality $f^*_{y^2} \le f^{**}_{y^2} = F^*_{y^2}$ and the assumption $\epsilon > 0$ imply that, for all$y^2 \in$

$$\left\|(1+\epsilon)^{-\frac{\epsilon}{(1+3\epsilon)(1+2\epsilon)}}\ell^{\epsilon+\frac{\epsilon^2(5+6\epsilon)}{(1+3\epsilon)(1+2\epsilon)}}(1+\epsilon)f^*(1+\epsilon)\right\|_{1+2\epsilon,(0,1)} \lesssim (\ln(y^2))^{\frac{1}{}}$$

t-hand side of (52) can be estimated from below by

$$\left(\int_{y^2}^{\omega^2} (1+\epsilon)^{-1}\ell^{-1}(1+\epsilon)d(1+\epsilon)\right)^{\frac{1}{1+2\epsilon}} \approx (\ln(y^2))^{\frac{1}{1+2\epsilon}} \text{ for all} y^2 \in \left(0,\frac{\omega^2}{2}\right.$$

that it must be $\epsilon \ge 0$.

slightly modify the approach of part (i). Now, we put $F_{y^2}(x^2) := f^*_{y^2}(V_n|x^2|^n)$,.

3.5(i) (with the expression on the second line of (8)) instead of Proposition 3.5(ii) a

$= f^*_{y^2}$.

4: Now we prove the necessity of the condition $\epsilon \ge 0$.

the contrary, suppose that $\epsilon > 0$.Hence, $0 \le \epsilon \le \infty$.

e (4) is assumed to hold for all functions from $B^{0,\frac{\epsilon+3\epsilon^2-1}{1+3\epsilon}}_{1+\epsilon,1+3\epsilon}$, Proposition 3.6 and Rema

$$+\epsilon)^{\frac{\epsilon}{(1+\epsilon)(1+2\epsilon)}}\ell^{\frac{\epsilon^2(5+6\epsilon)}{(1+\epsilon)(1+2\epsilon)}}(1+\epsilon)f^*(1+\epsilon)\Big\|_{1+2\epsilon,(0,1)}$$

$$\lesssim \left\|(1+\epsilon)^{\frac{1}{1+3\epsilon}}\ell^{\frac{\epsilon+3\epsilon^2-1}{1+3\epsilon}}(1+\epsilon)\left(\int_0^{1+\epsilon} f^*(u)^{1+\epsilon}du\right)^{\frac{1}{1+\epsilon}}\right\|_{1+3\epsilon,(0,1)}$$

$\mathcal{M}_0(\mathbb{R}^n)$ with $|\mathrm{supp}\, f\,|_n \le 1$. One can see that (53) remains true if we on

: 1. (Indeed, if $f \in \mathcal{M}_0(\mathbb{R}^n)$, take $f_1 := f^*(V_n|\cdot|^n)\chi_{[0,1)}(V_n|\cdot|^n)$. Consequer

or all $0 \le \epsilon < 1$, and $|\mathrm{supp}\, f_1|_n \le 1$. Thus, applying (53) to f_1, we obtain th

$f := |g|^{\frac{1}{1+\epsilon}}$. Then (53) yields

$$\frac{\epsilon}{(1+2\epsilon)(1+\epsilon)}\ell^{\epsilon(1+\epsilon)}(1-\epsilon)g^*(1-\epsilon)\Big\|_{\frac{1+2\epsilon}{1+\epsilon},(0,1)} \lesssim \left(\int_0^\infty v^2(1-\epsilon)g^{**}(1-\epsilon)^{\frac{1+3\epsilon}{1+\epsilon}}d(1\right.$$

$\mathcal{M}_0(\mathbb{R}^n)$(or even for any measurable function $g$on $\mathbb{R}^n$).

that $0 \le \epsilon < \infty$. Then (54) implies that the inequality

$$\left(\int_0^\infty w^2(1-\epsilon)g^{**}(1-\epsilon)^{\frac{1+2\epsilon}{1+\epsilon}}d(1-\epsilon)\right)^{\frac{1+\epsilon}{1+2\epsilon}} \lesssim \left(\int_0^\infty v^2(1-\epsilon)g^{**}(1-\epsilon)\ d(1\right.$$

measurable $g$on $\mathbb{R}^n$, where, for all $0 \le \epsilon < \infty$,

$$w^2(1+\epsilon) = (1+\epsilon)^{\frac{2\epsilon}{1+\epsilon}}\ell^{\epsilon(1+2\epsilon)}(1+\epsilon)\chi_{(0,1)}(1+\epsilon)$$

$$v^2(1+\epsilon) = (1+\epsilon)^{\frac{2\epsilon}{1+\epsilon}}\ell^{\epsilon+\epsilon^2-1}(1+\epsilon)\chi_{(0,1)}(1+\epsilon) + \chi_{[1,\infty)}(1+\epsilon).$$

on 4.8$\epsilon \ge 0$, inequality (55) holds only if

$$+\epsilon)^{\frac{(1+2\epsilon)}{\epsilon}} \sup_{y^2\in[1+\epsilon,1)} y^{-\frac{2(1+2\epsilon)}{\epsilon}}$$

$$\times \frac{\left(y^{\frac{2(1+2\epsilon)}{\epsilon}}\ell^{(\epsilon+\epsilon^2)\left(\frac{1+2\epsilon}{\epsilon}\right)}(y^2)\right)}{\left((1+\epsilon)\ \ell^{\epsilon+\epsilon^2-1}(1+\epsilon) + (1+\epsilon)\ \left(\int_{1+\epsilon}^1 s^{-1}\left(s^{\frac{2\epsilon}{1+\epsilon}}\ell^{\epsilon+\epsilon^2-1}(s)\right)ds + \int_1^\infty s\right.\right.}$$

for all $y^2 \in \left(0,\frac{\omega^2}{2}\right)$. Therefore $F_{y^2} \in B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon}$ and $\|F_{y^2}\|_{B^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}_{1+\epsilon,1+\epsilon}} \lesssim (\ln(y^2))^{\frac{1}{1+\epsilon}}$ for all $y^2 \in \left(0,\frac{\omega^2}{2}\right)$. This estimate, (4), the inequality $f^*_{y^2} \leq f^{**}_{y^2} = F^*_{y^2}$ and the assumption $\epsilon > 0$ imply that, for all$y^2 \in \left(0,\frac{\omega^2}{2}\right)$,

$$\left\|(1+\epsilon)^{-\frac{\epsilon}{(1+3\epsilon)(1+2\epsilon)}}\ell^{\epsilon+\frac{\epsilon^2(5+6\epsilon)}{(1+3\epsilon)(1+2\epsilon)}}(1+\epsilon)f^*(1+\epsilon)\right\|_{1+2\epsilon,(0,1)} \lesssim (\ln(y^2))^{\frac{1}{1+\epsilon}}. \tag{52}$$

Since the left-hand side of (52) can be estimated from below by

$$\left(\int_{y^2}^{\omega^2}(1+\epsilon)^{-1}\ell^{-1}(1+\epsilon)d(1+\epsilon)\right)^{\frac{1}{1+2\epsilon}} \approx (\ln(y^2))^{\frac{1}{1+2\epsilon}} \text{ for all} y^2 \in \left(0,\frac{\omega^2}{2}\right),$$

we conclude that it must be $\epsilon \geq 0$.

We slightly modify the approach of part (i). Now, we put $F_{y^2}(x^2) := f^*_{y^2}(V_n|x^2|^n), x^2 \in \mathbb{R}^n$, we apply Proposition 3.5(i) (with the expression on the second line of (8)) instead of Proposition 3.5(ii) and make use of the equality $F^*_{y^2} = f^*_{y^2}$.

Step 4: Now we prove the necessity of the condition $\epsilon \geq 0$.

On the contrary, suppose that $\epsilon > 0$.Hence, $0 \leq \epsilon \leq \infty$.

Since (4) is assumed to hold for all functions from $B^{0,\frac{\epsilon+3\epsilon^2-1}{1+3\epsilon}}_{1+\epsilon,1+3\epsilon}$, Proposition 3.6 and Remark 7.1 imply that

$$\left\|(1+\epsilon)^{\frac{\epsilon}{(1+\epsilon)(1+2\epsilon)}}\ell^{\frac{\epsilon^2(5+6\epsilon)}{(1+\epsilon)(1+2\epsilon)}}(1+\epsilon)f^*(1+\epsilon)\right\|_{1+2\epsilon,(0,1)}$$

$$\lesssim \left\|(1+\epsilon)^{\frac{1}{1+3\epsilon}}\ell^{\frac{\epsilon+3\epsilon^2-1}{1+3\epsilon}}(1+\epsilon)\left(\int_0^{1+\epsilon} f^*(u)^{1+\epsilon}du\right)^{\frac{1}{1+\epsilon}}\right\|_{1+3\epsilon,(0,1)} \tag{53}$$

for all $f \in \mathcal{M}_0(\mathbb{R}^n)$ with $|\text{supp}\, f|_n \leq 1$. One can see that (53) remains true if we omit the assumption $|\text{supp}\, f|_n \leq 1$. (Indeed, if $f \in \mathcal{M}_0(\mathbb{R}^n)$, take $f_1 := f^*(V_n|\cdot|^n)\chi_{[0,1)}(V_n|\cdot|^n)$. Consequently, $f_1^*(1-\epsilon) = f^*(1-\epsilon)$ for all $0 \leq \epsilon < 1$, and $|\text{supp}\, f_1|_n \leq 1$. Thus, applying (53) to f_1, we obtain the result.) Let $g \in \mathcal{M}_0(\mathbb{R}^n)$and $f := |g|^{\frac{1}{1+\epsilon}}$. Then (53) yields

$$\left\|(1-\epsilon)^{\frac{\epsilon}{(1+2\epsilon)(1+\epsilon)}}\ell^{\epsilon(1+\epsilon)}(1-\epsilon)g^*(1-\epsilon)\right\|_{\frac{1+2\epsilon}{1+\epsilon},(0,1)} \lesssim \left(\int_0^\infty v^2(1-\epsilon)g^{**}(1-\epsilon)^{\frac{1+3\epsilon}{1+\epsilon}}d(1+\epsilon)\right)^{\frac{1+\epsilon}{1+3\epsilon}} \tag{54}$$

for all $g \in \mathcal{M}_0(\mathbb{R}^n)$(or even for any measurable function $g$on $\mathbb{R}^n$).

Assume first that $0 \leq \epsilon < \infty$. Then (54) implies that the inequality

$$\left(\int_0^\infty w^2(1-\epsilon)g^{**}(1-\epsilon)^{\frac{1+2\epsilon}{1+\epsilon}}d(1-\epsilon)\right)^{\frac{1+\epsilon}{1+2\epsilon}} \lesssim \left(\int_0^\infty v^2(1-\epsilon)g^{**}(1-\epsilon)\ d(1-\epsilon)\right) \tag{55}$$

holds for all measurable $g$on $\mathbb{R}^n$, where, for all $0 \leq \epsilon < \infty$,

$$w^2(1+\epsilon) = (1+\epsilon)^{\frac{2\epsilon}{1+\epsilon}}\ell^{\epsilon(1+2\epsilon)}(1+\epsilon)\chi_{(0,1)}(1+\epsilon)$$

and

$$v^2(1+\epsilon) = (1+\epsilon)^{\frac{2\epsilon}{1+\epsilon}}\ell^{\epsilon+\epsilon^2-1}(1+\epsilon)\chi_{(0,1)}(1+\epsilon) + \chi_{[1,\infty)}(1+\epsilon).$$

By Proposition 4.8$\epsilon \geq 0$, inequality (55) holds only if

$$\infty > \int_0^1 (1+\epsilon)^{\frac{(1+2\epsilon)}{\epsilon}} \sup_{y^2\in[1+\epsilon,1)} y^{-\frac{2(1+2\epsilon)}{\epsilon}}$$

$$\times \frac{\left(y^{\frac{2(1+2\epsilon)}{\epsilon}}\ell^{(\epsilon+\epsilon^2)\left(\frac{1+2\epsilon}{\epsilon}\right)}(y^2)\right)}{\left((1+\epsilon)\ \ell^{\epsilon+\epsilon^2-1}(1+\epsilon) + (1+\epsilon)\ \left(\int_{1+\epsilon}^1 s^{-1}\left(s^{\frac{2\epsilon}{1+\epsilon}}\ell^{\epsilon+\epsilon^2-1}(s)\right)ds + \int_1^\infty s^{-1}\,ds\right)\right)^{\frac{1+4\epsilon}{\epsilon}}}$$

$$\times (1+\epsilon)^{\frac{1+3\epsilon}{1+\epsilon}}\ell^{\epsilon+3\epsilon^2-1}(1+\epsilon)\int_{1+\epsilon}^1 s^{-\frac{1+3\epsilon}{1+\epsilon}}\left(s^{\frac{2\epsilon}{1+\epsilon}}\ell^{\epsilon+3\epsilon^2-1}(s)\right)ds\,(1+\epsilon)^{\frac{2\epsilon}{1+\epsilon}}d(1+\epsilon) =: I.$$

However,

$$\frac{((1+\epsilon)(1+2\epsilon))(1+\epsilon)\ell^{\epsilon+\epsilon^2-1}(1+\epsilon)\ell^{\epsilon+3\epsilon^2}((1+\epsilon))(1+\epsilon)^{-1}}{\left(\ell^{\epsilon+\epsilon^2-1}(1+\epsilon)+\left(\ell^{\epsilon+\epsilon^2}(1+\epsilon)+\frac{1+\epsilon}{2\epsilon}\right)\right)^{\frac{1+4\epsilon}{\epsilon}}} \approx \int_0^{\frac{1}{2}} (1+\epsilon)^{-1}\ell^{-}$$

ntradiction. Consequently, $\epsilon \geq 0$.

that $\epsilon = \infty$. Therefore, $\epsilon > 0$. Inequality (54) implies that

$$\left(\int_0^{\infty} w^2(1+\epsilon)g^*(1+\epsilon)^{\frac{1+2\epsilon}{1+\epsilon}}\, d(1+\epsilon)\right)^{\frac{1+\epsilon}{1+2\epsilon}} \lesssim ess \sup_{0\leq\epsilon<\infty} v^2(1+\epsilon)g^{**}(1+\epsilon)$$

rable g in $\mathbb{R}^n$, where, for all $0 \leq \epsilon < \infty$,

$$w^2(1+\epsilon) = (1+\epsilon)^{\frac{2\epsilon}{1+\epsilon}}\ell^{\frac{(\epsilon+\epsilon^2-1)(1+2\epsilon)}{1+\epsilon}}(1+\epsilon)\chi_{(0,1)}(1+\epsilon)$$

$$v^2(1+\epsilon) = (1+\epsilon)\ell^{\frac{(\epsilon+3\epsilon^2-1)(1+\epsilon)}{1+3\epsilon}}(1+\epsilon)\chi_{(0,1)}(1+\epsilon) + \ell(1+\epsilon)\chi_{[1,\infty]}(1+$$

measure on $[0,\infty)$which is absolutely continuous with respect to the Lebesgue me

$$dv^2(1+\epsilon) = \begin{cases} (1-\epsilon)^{-1}\ell^{-\frac{(\epsilon+\epsilon^2-1)(1+2\epsilon)}{1+\epsilon}-1}(1-\epsilon)\, d(1+\epsilon) & \text{if } 0 \leq \epsilon < 1, \\ (1+\epsilon)^{\frac{1+2\epsilon}{\epsilon}}\ell^{-\frac{1+2\epsilon}{\epsilon}}(1+\epsilon)\, d(1+\epsilon) & \text{if } \epsilon > 0. \end{cases}$$

$$I \gtrsim \int_0^1 \left(\sup_{s\in(1+\epsilon,1)} \ell^{\frac{(\epsilon+\epsilon^2-1)(1+2\epsilon)}{1+\epsilon}}(s)\right)(1+\epsilon)^{-1}\ell^{-\frac{(\epsilon+\epsilon^2-1)(1+2\epsilon)}{1+\epsilon}-1}(1+\epsilon)\, d(1-$$

$$\approx \int_0^1 (1+\epsilon)^{-1}\ell^{-1}(1+\epsilon)\, d(1+\epsilon) = \infty,$$

ntradiction. Consequently, $\epsilon \geq 0$.

Theorem 3.2

ew of Theorem 3.1, the sufficiency of the condition that $\kappa$is bounded is obvious. T n is also necessary (see [17]).

1: Assume $\epsilon \geq 0$. Take $y^2 \in (0,1/2)$and $f_{y^2} \in L_{1+\epsilon}(\mathbb{R}^n)$with $f^*_{y^2} = \chi[0,y^2)$.It i

$$+\epsilon)\left(\int_{(1+\epsilon)^n}^{\infty} s^{-\frac{1+\epsilon}{n}}\int_0^s \left(f^*_{y^2}(u)-f^*_{y^2}(s)\right)^{1+\epsilon} du\frac{ds}{s}\right)^{\frac{1}{1+\epsilon}} \approx \min \left\{y^{\frac{2}{1+\epsilon}},(1+\right.$$

and $y^2 \in (0,1/2)$.

Case $0 < \epsilon < \infty$.

$F_{y^2}(x^2) = f^{**}_{y^2}(V_n|x^2|^n), x^2 \in \mathbb{R}^n$, we get $\|F_{y^2}\|_{1+\epsilon} = \|F^*_{y^2}\|_{1+\epsilon} = \|f^{**}_{y^2}\|_{1}$

ally$^2 \in (0,1/2)$. Moreover, Proposition 3.5(ii) and (57) imply that ω

$y^{\frac{2}{1+\epsilon}},(1+\epsilon)y^{2\left(\frac{1}{1+\epsilon}-\frac{1}{n}\right)}\}$ for all $y^2 \in (0,1/2)$and $\epsilon \geq 0$. Hence,

$$\left\|(1+\epsilon)^{-\frac{1}{1+3\epsilon}}\ell^{\frac{\epsilon+3\epsilon^2-1}{1+3\epsilon}}(1+\epsilon)\omega_1^2(F_{y^2},1+\epsilon)_{1+\epsilon}\right\|_{1+3\epsilon,(0,1)}$$

$$\lesssim y^{2\left(\frac{1}{1+\epsilon}-\frac{1}{n}\right)}\left\|(1+\epsilon)^{\frac{2\epsilon}{1+3\epsilon}}\ell^{\frac{\epsilon+3\epsilon^2-1}{1+3\epsilon}}(1+\epsilon)\right\|_{1+3\epsilon,\left(0,y^{\frac{2}{n}}\right)}$$

$$+\left\|y^{\frac{2}{1+\epsilon}}(1+\epsilon)^{-\frac{1}{1+3\epsilon}}\ell^{\frac{\epsilon+3\epsilon^2-1}{1+3\epsilon}}(1+\epsilon)\right\|_{1+3\epsilon,\left(y^{\frac{2}{n}},1\right)}$$

$^2 \in (0,1/2)$. Therefore, $F_{y^2} \in B^{0,\frac{\epsilon+3\epsilon^2-1}{1+3\epsilon}}_{1+\epsilon,1+3\epsilon}$and

$$\|F_{y^2}\|_{B^{0,\frac{\epsilon+3\epsilon^2-1}{1+3\epsilon}}_{1+\epsilon,1+3\epsilon}} \lesssim y^{\frac{2}{1+\epsilon}}\ell^{\epsilon}(y^2) \text{ for all} y^2 \in (0,1/2).$$

mate, (5), the inequality $f^*_{y^2} \leq f^{**}_{y^2} = F^*_{y^2}$and the assumption $\epsilon \geq 0$imply that

$$\left\|(1+\epsilon)^{\frac{\epsilon}{(1+2\epsilon)(1+\epsilon)}}\ell^{\epsilon}(1+\epsilon)\kappa(1+\epsilon)\right\|_{1+2\epsilon,(0,y^2)} \lesssim y^{\frac{2}{1+\epsilon}}\ell^{\epsilon}(y^2).$$

$$\kappa(y^2)y^{\frac{2}{1+\epsilon}}\ell^{\epsilon}(y^2) \lesssim y^{\frac{2}{1+\epsilon}}\ell^{\epsilon}(y^2) \text{ for all} y^2 \in (0,1/2).$$

Hence, κmust be bounded.

(ii) Case $\epsilon = 0$.

Defining $F_{y^2}(x^2) = f_{y^2}^*(V_n|x^2|^n), x^2 \in \mathbb{R}^n$, we get $\|F_{y^2}\|_1 = \|F_{y^2}^*\|_1 = \|f_{y^2}^*\|_1 = y^2$. Moreover, Proposition 3.5(i) and (57) yield $\omega_1^2(F_{y^2}, 1+\epsilon)_1 \lesssim \min \{y^2, (1+\epsilon)y^{2(1-\frac{1}{n})}\}$ for all $y^2 \in (0,1/2)$and $\epsilon \geq 0$. The rest follows essentially as in part (i) (now with $\epsilon = 0$ and $F_{y^2}^* = f_{y^2}^*$).

Step 2: Assume now that $0 \leq \epsilon < \infty$. In particular, $\epsilon > 0$.

For any given $y^2 \in (0,1/2)$, put $f_{y^2}(x^2) := y^{-\frac{2}{1+2\epsilon}}\ell^{\frac{\epsilon}{(1+2\epsilon)(1+2\epsilon)}}(y^2)\chi_{[0,y^2]}(V_n|x^2|^n) + (V_n|x^2|^n)^{-\frac{1}{1+3\epsilon}}\ell^{\frac{\epsilon}{(1+2\epsilon)(1+2\epsilon)}}(V_n|x^2|^n)\chi_{(y^2,1)}(V_n|x^2|^n), x^2 \in \mathbb{R}^n$. Then

$$f_{y^2}^*(1+\epsilon) = y^{-\frac{2}{1+3\epsilon}}\ell^{\frac{\epsilon}{(1+2\epsilon)(1+2\epsilon)}}(y^2)\chi_{(0,y^2]}(1+\epsilon) + (1+\epsilon)^{-\frac{1}{1+3\epsilon}}\ell^{\frac{\epsilon}{(1+2\epsilon)(1+2\epsilon)}}(1+\epsilon)\chi_{(y^2,1)}(1+\epsilon),$$
$$\epsilon \geq 0.$$

We proceed as in part (i) of Step 3 of the proof of Theorem 3.1. Defining $F_{y^2}(x^2) = f_{y^2}^{**}(V_n|x^2|^n), x^2 \in \mathbb{R}^n$, we see that$\|F_{y^2}\|_{1+3\epsilon} \lesssim \ell^{\frac{1}{1+2\epsilon}}(y^2)$for all$y^2 \in (0,1/2)$. Moreover, we obtain (49), where now

$$A \lesssim \ell^{\frac{\epsilon+2\epsilon^2-1}{1+2\epsilon}}(y^2) \text{ and } B \lesssim \ell^{\frac{\epsilon+2\epsilon^2-1}{1+2\epsilon}}(y^2)$$

for all $y^2 \in (0,1/2)$. Therefore, $F_{y^2} \in B_{1+3\epsilon,1+\epsilon}^{0,\frac{\epsilon+3\epsilon^2-1}{1+3\epsilon}}$and $\|F_{y^2}\|_{B_{1+3\epsilon,1+\epsilon}^{0,\frac{\epsilon+3\epsilon^2-1}{1+3\epsilon}}} \lesssim \ell^{\frac{\epsilon+2\epsilon^2-1}{1+2\epsilon}}(y^2)$for all $y^2 \in (0,1/2)$. This estimate, (5), the inequality $f_{y^2}^* \leq f_{y^2}^{**} = F_{y^2}^*$and the assumption $\epsilon > 0$imply that

$$\left\|(1+\epsilon)^{\frac{\epsilon}{(1+2\epsilon)(1+\epsilon)}}\ell^{\frac{3\epsilon}{1+\epsilon}}(1+\epsilon)\kappa(1+\epsilon)f_{y^2}^*(1+\epsilon)\right\|_{1+\epsilon,(y^2,y)} \lesssim \ell^{\frac{\epsilon+2\epsilon^2-1}{1+2\epsilon}}(y^2)$$

for all $y^2 \in (0,1/2)$. Since the left-hand side of the last expression can be estimated from below by

$$\kappa(y)\left\|(1+\epsilon)^{-\frac{1}{1+\epsilon}}\ell^{\epsilon}(1+\epsilon)\right\|_{1+2\epsilon,(y^2,y)} \approx \kappa(y)\ell^{\frac{\epsilon+2\epsilon^2-1}{1+2\epsilon}}(y^2) \text{ for all} y^2 \in (0,1/2),$$

we conclude that κmust be bounded.

9. Proof of Theorem 3.3 (see [17])

We refer only to the case $0 < \epsilon < \infty$; the case $\epsilon = 0$ can be easily adapted.

Put $A := B_{1+\epsilon,1+\epsilon}^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}$. By Theorem 3.1 with $\epsilon = \infty$,

$$(1+\epsilon)^{\frac{1}{1+\epsilon}}\ell^{\epsilon}(1+\epsilon)f^*(1+\epsilon) \lesssim 1$$

for all $0 < \epsilon < 1$ and $f \in A$ with $\|f\|_A \leq 1$. Therefore,

$$\sup_{\|f\|_A \leq 1} f^*(1-\epsilon) \lesssim (1-\epsilon)^{-\frac{1}{1+\epsilon}}\ell^{-\epsilon}(1-\epsilon) \text{for all } 0 < \epsilon < 1. \tag{59}$$

On the other hand, consider the functions $F_{y^2}, y^2 \in (0,1/2)$, from Step 1 of the proof of Theorem 3.2. By (58), there exists c > 0 such that

$$\|F_{y^2}\|_A \leq c y^{\frac{2}{1+\epsilon}}\ell^{\epsilon}(y^2) \text{ for all} y^2 \in (0,1/2).$$

Together with the inequality $F_{y^2}^* \equiv f_{y^2}^{**} \geq f_{y^2}^* \equiv \chi_{[0,y^2)}$, this implies that

$$\sup_{\|f\|_A \leq 1} f^*(1+\epsilon) \geq c^{-1}y^{-\frac{2}{1+\epsilon}}\ell^{-\epsilon}(y^2)\chi_{[0,y^2)}(1+\epsilon) \tag{60}$$

for all $\epsilon \geq 0$ and $y^2 \in (0,1/2)$. Thus, taking$y^2 = 2\left(\frac{1}{4}-\epsilon\right)$ for every $0 < \epsilon < \frac{1}{4}$, we obtain from (60) that

$$\sup_{\|f\|_A \leq 1} f^*\left(\frac{1}{4}-\epsilon\right) \geq c^{-1}\left(2\left(\frac{1}{4}-\epsilon\right)\right)^{-\frac{1}{1+\epsilon}}\ell^{-\epsilon}\left(2\left(\frac{1}{4}-\epsilon\right)\right)\chi_{\left[0,2\left(\frac{1}{4}-\epsilon\right)\right)}\left(\frac{1}{4}-\epsilon\right) \approx \left(\frac{1}{4}-\epsilon\right)^{-\frac{1}{1+\epsilon}}\ell^{-\epsilon}\left(\frac{1}{4}-\epsilon\right)$$

for all $0 < \epsilon < \frac{1}{4}$. Together with (59), this gives

$$\sup_{\|f\|_A \leq 1} f^*(1+\epsilon) \approx (1+\epsilon)^{-\frac{1}{1+\epsilon}}\ell^{-\epsilon}(1+\epsilon) =: h(1+\epsilon) \text{for all small } \epsilon \geq 0.$$

Since the function h is positive, continuous and non-increasing on some $(0,\varepsilon], \varepsilon \in (0,1/2)$, and $\lim_{1+\epsilon\to 0} +$ $h(1+\epsilon) = \infty$, this function h is a growth envelope function of the space $A = B_{1+\epsilon,1+\epsilon}^{0,\frac{\epsilon+\epsilon^2-1}{1+\epsilon}}$.

As to the fine index, notice that $H(1+\epsilon) := -\ln\ h(1+\epsilon)$ satisfies $H'(1+\epsilon) \approx \frac{1}{1+\epsilon}$ on some small interval $(0,\varepsilon)$. Therefore, $d\mu_H(1+\epsilon) \approx \frac{1}{1+\epsilon}\ d(1+\epsilon)$ and Theorem 3.1 implies that (with the usual modification in the case $\epsilon = \infty$) whenever $1+2\epsilon \in [\max \{1+\epsilon, 1+\epsilon\}, \infty]$. On the other hand, it is also possible to prove that this cannot hold for $1+2\epsilon \in (0, \max \{1+\epsilon, 1+\epsilon\})$.

In order to see this, we shall show first that if (61) holds then it must be $\epsilon \geq 0$. We follow thesame construction as in the proof of Step 3 of Theorem 3.1, now with $\omega^2 \in (0, \varepsilon]$. Since we use (61) instead of (4), now the counterpart of (52) reads as

$$\left\| (1+\epsilon)^{\frac{\epsilon}{(1+\epsilon)(1+2\epsilon)}} \ell^{\epsilon}(1+\epsilon) f^*(1+\epsilon) \right\|_{1+2\epsilon,(0,\varepsilon)} \lesssim (\ln \quad \ell(y^2))^{\frac{1}{1+\epsilon}} \text{ for all} y^2 \in \left(0, \frac{\omega^2}{2}\right). \qquad (62)$$

If we assumed that $\epsilon > 0$, then the left-hand side of (62) could be estimated from below by

$$\left(\int_{y^2}^{\omega^2} (1+\epsilon)^{-1} \ell^{-\frac{1+2\epsilon}{1+3\epsilon}} (1+\epsilon) d(1+\epsilon) \right)^{\frac{1}{1+2\epsilon}} \approx \ell^{\frac{\epsilon}{(1+2\epsilon)(1+3\epsilon)}}(y^2) \text{ for all} y^2 \in \left(0, \frac{\omega^2}{2}\right),$$

and we would get a contradiction.

So, we have just shown that (61) implies $\epsilon \geq 0$. Consequently, $\frac{1}{\max \{1+\epsilon, 1+2\epsilon\}} - \frac{1}{1+2\epsilon} = 0$ and we can now use Theorem 3.1 to show that $\epsilon \geq 0$.

Therefore, (61) holds if and only if $1+2\epsilon \in [\max \{1+\epsilon, 1+\epsilon\}, \infty]$.

Conclusion

We study mainly the worst-case in a Soboleve space for cubature and extremal systems of points with numerical integration over the sphere . An approximation of the constructive polynomial on the sphere was considered. We find the optimal lower bounds for cubature error in Sobolev spaces of arbitrary order. The quadrature in Besov spaces on the Euclidion sphere was shown. We obtain the sobolev error estimates and determined the Bernsten inequality for scattered data interpolation, we also establish the spherical basis functions and construct the uniform distribution of points on spheres. We gine the structure of the orthogonal ,inequalities and orthonormal polynomials with exponential-type weights.we investigate the sharp embeddings of Besov-type spaces involving only logarithmic smoothness.

درسنا الي حد بعيد الحالة الاسوأ في اطار فضاء سوبوليف للتكعيب والانظمة العظمي للنقاط طبقا للتكامل العددي فوق الكرة . تم ايجاد الحديات الادني والامثل لخطا التكعيب في فضاءات سوبوليف للرتبة الاختيارية . اوضحنا التربيع في فضاءات بيسوف علي الكرة الاقليدية . تم اعطاء تقديرات خطا سوبوليف وتحديد متباينه بير نشتاين لاستكمال البيانات والتبعثر . ايضا اسست دوال الاساس الكروي وبناء التوزيع المنتظم للنقاط علي الكرات . تم اعطاء تشييد كثيرات الحدود المتعامدة للمتباينات والمنتظمة المتعامدة طبقا للمرجحات اسية النوع . وتقصينا الغمر القاطع للفضاءات نوع بيسوف المتضمنه الملسان اللوغاريثمي فقط.

List of Symbols

Symbols

$H^{\frac{3}{2}}$: *Hubert space*

H^{s} : *Sobolev space*

sup: *supremum*

L_2 : *Hilbert space*

POL : *polynomial*

avg : *average*

min : *minimum*

inf : *infimum*

max : *maximum*

det : *determinant*

$supp$: *support*

H^2 : *Hihbert space*

L^p : *Lebesgue space*

$\oplus$: *Direct sum*

ess: *essential*

dim : *dimension*

$dist$: *distant*

RBF : *Radial Basis Function*

$RKHS$: *Reproducing Kernel Hilbert Space*

$SBFs$: *Spherical Basis Functions*

W_2^2 : *Sobolev space*

vol : *volume*

int : *interior*

$a.e$: *almost every where*

MRS : *Mhaskov – Rahmanov – Saff*

deg : *degree*

$B_{p,r}^{\sigma,b}$: *Besove spaces*

L^{∞} : *Lebesgue space*

$L_{p,q,y}^{loc}$: *Lorentz – Zygmund spaces*

Loc : *Locally*

$H^{\frac{3}{2}}$: ***Hubert space***

H^{s} : ***Sobolev space***

sup: ***supremum***

L_2 : ***Hilbert space***

POL : ***polynomial***

avg : ***average***

min : ***minimum***

inf : ***infimum***

max : ***maximum***

det : ***determinant***

supp : ***support***

H^{2} : ***Hihbert space***

L^{p} : ***Lebesgue space***

$\oplus$: ***Direct sum***

ess: ***essential***

dim : ***dimension***

dist : ***distant***

RBF : ***Radial Basis Function***

RKHS : ***Reproducing Kernel Hilbert Space***

SBFs : ***Spherical Basis Functions***

W_2^2 : ***Sobolev space***

vol : ***volume***

int : ***interior***

a.e : ***almost every where***

MRS : ***Mhaskov − Rahmanov − Saff***

deg : ***degree***

$B_{p,r}^{\sigma,b}$: ***Besove spaces***

L^{∞} : ***Lebesgue space***

$L_{p,q,y}^{loc}$: ***Lorentz − Zygmund spaces***

Loc : ***Locally***

References

[1] A. Caetano, H.-G. Leopold, Local growth envelopes of Triebel-Lizorkin spaces of gener
J. Fourier Anal. Appl. 12 (2006) 427−445.

[2] A.M. Caetano,W. Farkas, Local growth envelopes of Besov spaces of generalized sm
Anwendungen 25 (2006) 265−298.

[3] A.M. Caetano, S.D. Moura, Local growth envelopes of spaces of generalized smoothne
Math. Inequality Appl. 7 (2004) 573−606.

[4] A.M. Caetano, S.D. Moura, Local growth envelopes of spaces of generalized smoothn
case, Math. Nachr. 273 (1) (2004) 43−57.

[5] A. Gogatishvili, L. Pick, Discretization and anti-discretization of rearrangement-inva
Mat. 47 (2) (2003) 311−358.

[6] A. Gogatishvili, L. Pick, Embeddings and duality theorems for weak classical Lore
Math. Bull. 49 (1) (2006) 82−95.

[7] M. Goldman, Embeddings of Nikol'skij−Besov spaces into weighted Lorents space
Steklova 180 (1987) 93−95 (in Russian).

[8] M.L. Gol'dman, R. Kerman, On optimal embedding of Calderón spaces and general
Trudy Mat. Inst. Steklova 243 (2003) 161−193 (in Russian) (English translation: Proc.
243 (2003) 154−184).

[9] P. Gurka, B. Opic, Sharp embeddings of Besov spaces with logarithmic smoo
Complutense 18(2005) 81−110.

[10] P. Gurka, B. Opic, Sharp embeddings of Besov-type spaces, J. Comput. Appl. Math. 208

[11] G. Hardy, J.E. Littlewood, G. Pólya, Inequalities, reprinted second ed., Cambridge
Cambridge, 1994.

[12] D.D. Haroske, Envelopes in function spaces−a first approach, preprint Math/Inf/16/01,
Germany, 2001 (cf. also D.D. Haroske, Envelopes and sharp embeddings of function s
Hall, CRC, BocaRaton, 2007).

[13] V. Kolyada, Estimates of rearrangements and embedding theorems, Mat. Sb. 136 (19
transl. in Math. USSR-Sb. 64(1) (1989) 1−21).

[14] Yu. Netrusov, Theorems for the embedding of Besov spaces into ideal spaces, Z
Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 159 (1987) 69−82 (Russian) (English tra
Math. 47 (1989) 2871−2881).

[15] B. Opic, A. Kufner, Hardy-type Inequalities, Pitman Research Notes in Mathematics,
Science & Technology, Harlow, 1990.

[16] H. Triebel, The Structure of Functions, Birkhäuser, Basel, 2001.

[17] António M. Caetano, Amiran Gogatishvili, Bohumir Opic, Sharp embeddings of Beso
only logarithmic smoothness, Journal of Approximation Theory 152 (2008) 188 − 214.