Some Aspects of apply the Hankel transform method in many different areas of physics and Engineering

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Abstract

We establish a new solution of ordinary differential equation (ODEs) and partial differential equation (PDEs) using Hankel transform methods and compare the new solutions with other methods, This method very efficient, easy and not wide use in solving some problem in ordinary differential equations (ODEs). and Partial differential equations (PDEs) compared with the other methods , apply the Hankel transform method in many different areas of physics and engineering. In this paper we used some mathematics program (Matlab)Results: we found Hankel transform is very useful for solving solution of ordinary differential equation (ODEs) and partial differential equation (PDEs) using Hankel transform methods and compare the new solutions with other methods, apply the Hankel transform method in many different areas of physics and engineering

Keywords: Hankel transform, Bessel transform, Ordinary Differential Equation , Partial Differential Equation , Bessel functions

مستخلص:

الهدف من هذه الدراسة هو أن نؤسس حلاً جديدًا حلاً جديدًا للمعادلة التفاضلية العادية (ODEs) والمعادلة التفاضلية الجزئية (PDEs) باستخدام طرق تحويل Hankel ومقارنة الحلول الجديدة بالطرق الأخرى ، هذه الطريقة فعالة جدًا وسهلة الاستخدام وليست على نطاق واسع في حل بعض المشكلات في المعادلات التفاضلية العادية (ODEs) والمعادلات التفاضلية الجزئية (PDEs) مقارنة بالطرق الأخرى ، يتم تطبيق طريقة تحويل هانكل في العديد من مجالات الفيزياء والهندسة المختلفة. في هذا البحث استخدمنا بعض برامج الرياضيات (ماتلاب).النتائج: وجدنا تحويل هانكل مفيدًا جدًا في حل المعادلة التفاضلية العادية (ODEs) والمعادلة التفاضلية الجزئية (PDEs) باستخدام طرق تحويل هانكل ومقارنة الحلول الجديدة بالطرق الأخرى ، وتطبيق طريقة

تحويـل هانـكل في العديـد مـن مجـالات الفيزيـاء المختلفـة والهندسـة.

الكلمات المفتاحية:تحويـل هانـكل ، تحويـل بيسـل ، معادلـة تفاضليـة عاديـة ، معادلـة تفاضليـة جزئيـة ، دوال بيسـل

Introduction:

In mathematics, the Hankel transform expresses any given function f(r) as the weighted sum of an infinite number of Bessel functions of the first kind Jν(kr). The Bessel functions in the sum are all of the same order ν, but differ in a scaling factor k along the r axis. ... It is also known as the Fourier–Bessel transform[1].

In 1960 B.R.Bhonsle obtain a relation between the Laplace transform of $t^{\mu}f(t)$ and the Hankel transform of $f(t)$,when $\mathcal{R}(\mu) > -1$.The result is stated in the form of a theorem[2]:

if f and $\mathcal{H}_{\nu}\{f;\xi\}$belong to $L(0,\infty)$and if $\mathcal{R}(\mu) > -1$,$\mathcal{R}(\mu+\nu) >$ ( )

$-1, \mathcal{R}$ p > 0 then :

$$\mathcal{L}\{t^{\mu}f(t);p\} = \int_0^{\infty} k(p,\xi)\mathcal{H}_{\nu}\{f;\xi\}d\xi$$

Where:

$$k(p,\xi) = \Gamma(\mu+v+1)\xi(p^2+\xi^2)^{-\frac{1}{2}(\mu+1)}P_{\mu}^{-v}\left(\frac{p}{\sqrt{p^2+\xi^2}}\right)$$

Proof. Since $f \in L(0,\infty)$ we have, by the Hankel transform inversion theorem. [3]

In 1974 J.R.RIDENHOUR and R.P.SONT assume : let $F_{\nu}(x)$, the Hankel transform of f , defined by $F_{\nu}(x) = \int_0^{\infty} f(t)\sqrt{xt}\ J_{\nu}(xt)dt$. It is proved that

the Parseval relation $\int_0^{\infty} F_{\nu}(x)\,G_{\nu}(x)dx = \int_0^{\infty} f(x)g(x)dx$ holds , if (i)$x^{\nu+\frac{1}{2}} f(x) \in$

$L(0,R)$ for each finite $R > 0$, f is bounded variation in $[a,\infty)$ for some $a > 0$, and

$f(x) \to 0$ as $x \to \infty$. (ii)$g \in L(0,\infty)$, g is of bounded variation in a neighborhood of every point where f is not and $G_{\nu}(x) = O(x^{-\lambda-\nu})$ as $x \to \infty$ for some $\lambda \geq 3/2$. [4]

In 1983 ROB BISSELINC AND RONNIE KOSLOFF, A new method is presented for the solution of the time dependent Schrödinger equation, expressed in polar or spherical coordinates. The radial part of the Laplacian operator is computed using a Fast Hankel Transform. An algorithm for the FHT is described, based on the Fast Fourier Transform. The accuracy of the Hankel method is checked for the two- and three-dimensional harmonic oscillator by comparing with the analytical solution. The Hankel method is applied to the system $H + H_2$, with Delves hyperspherical coordinates and is compared to the Fourier method. (5)

In 1997 I.ALI and S.KALLA , introduce a generalized form of the Hankel transform , and study some of its properties. A partial differential equation
associated with the problem of transport of a heavy pollutant (dust) from the ground level sources within the framework of the diffusion theory is treated by this integral transform. The pollutant concentration is expressed in terms of a given flux of dust from the ground surface to the atmosphere. (6)

1.2 Problem Statement:

In this research , we will use Hankel transform method to find the solution of partial differential equations , and apply relationship between Hankel transform and Fourier transform and applications .

1.3 Objective:

We establish a new solution of ordinary differential equation (ODEs) and partial differential equation (PDEs) using Hankel transform methods and compare the new solutions with other methods.

1.4 Research Methodology:

The method that we use to solve some problem in differential equations as follows

1. We apply the Hankel transform method in many different areas of physics
   And engineering

Applications

The Electrified Disc

Let v be the electric potential due to a flat circular electrified disc, with radius
$R = 1$, the center of the disc being at the origin of the three-dimensional space and its axis along the $z - axis$.
In polar coordinates, the potential satisfies [7]
Laplace's equation

$$\nabla^2 \equiv \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} = 0 \qquad (1)$$

The boundary conditions are

$$v(r,0) = v_0 \qquad ,0 \leq r < 1 \qquad (2)$$

$$\frac{\partial v}{\partial z}(r,0) = 0 \qquad , \qquad r > 1 \qquad (3)$$

In (3.9.2), v_0 is the potential of the disc. Condition (3.9.3) arises from the
symmetry about the plane $z = 0$.
Let
$V(s,z) = \mathcal{H}_0\{v(r,z)\}$

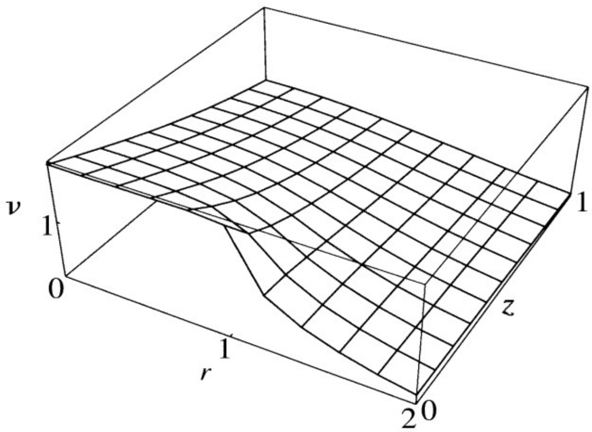


FIGURE 3.1 Electrical potential due to an electrified disc
So that

$$\mathcal{H}_0\{\nabla^2 v\} = -s^2\, V(s,z) + \frac{\partial^2 V}{\partial z^2}(s,z) = 0$$

the solution of this differential equation is

$$V(s,z) = A(s)e^{-sz} + B(s)e^{sz}$$

where A and B are functions that we have to determine using the boundary conditions. Because the potential vanishes as z tends to infinity, we have $B(s) \equiv 0$. By inverting the Hankel transform, we have [8]

$$v(r,z) = \int_0^\infty sA(s)\, e^{-sz} J_0(sr)ds \qquad (4)$$

The boundary conditions are now

$$v(r,0) = \int_0^\infty sA\,(s) J_0(rs)ds = v_0 \quad , \qquad 0 \le r < 0$$

(5)

$$\frac{\partial v}{\partial z}(r,0) = \int_0^\infty s^2 A(s) J_0(rs)ds = 0 \quad , \qquad r > 1$$

(6)

Using entries (8) and (9) of Table 9.1 , we see that $A(s) = \sin s/s^2$ so that

$$v(r,z) = \frac{2v}{\pi}\int_0^\infty \frac{\sin s}{s}\, e^{-sz} J_0(rs)ds \qquad (7)$$

In Figure 3.1 , the graphical representation of $v(r,z)$ $v_0 = 1$ for is depicted on the domain $0 \le r \le 2{,}0 \le z \le 1$. The evaluation of $v(r,z)$ requires numerical integration .

Equations(4) and (5) are special cases of the more general pair of equations [9]

$$\int_0^\infty f(t) t^{2\alpha} J_v(xt)dt = a(x) \quad , \qquad 0 \le x < 1 \qquad (8)$$

$$\int_0^\infty f(t)J_v(xt)dt = 0 \quad , \quad x > 1 \tag{9}$$

Where $a(x)$ *is given and* $f(x)$ is to be determined .
The solution of (3.9.3) can be expressed as a repeated integral .
(10)

$$f(x) = \frac{2^{-\alpha}x^{1-\alpha}}{\Gamma(\alpha+1)}\int_0^1 s^{-v-\alpha}J_{v+\alpha}(xs)\frac{d}{ds}\int_0^s a(t)t^{v+1}(s^2-t^2)^\alpha dtds \ , -1 < \alpha < 0 \tag{3.9.9}$$

$$f(x) = \frac{(2x)^{1-\alpha}}{\Gamma(\alpha)}\int_0^1 s^{-v-\alpha+1}J_{v+\alpha}(xs)\frac{d}{ds}\int_0^s a(t)t^{v+1}(s^2-t^2)^\alpha dtds \ , 0 < \alpha < 1 \tag{3.9.10}$$

If $a(x) = x^\beta$, *and* $\alpha < 1 , 2\alpha + \beta > -3/2 , \alpha + v > -1 , v > -1$ then

$$f(x) = \frac{\Gamma\left(1+\frac{\beta+v}{2}\right)x^{-(2\alpha+\beta+1)}}{2^\alpha\Gamma\left(1+\alpha+\frac{\beta+v}{2}\right)}\int_0^x t^{\alpha+\beta+1}J_{v+\alpha}(t)dt \tag{10}$$

With $B = v$ *and* $a < 1 , a + v > -1, v > -1$ further simplification is possible:

$$f(\mathrm{x}) = \frac{\Gamma(\mathrm{v}+1)}{(2\mathrm{x})^\alpha\Gamma(\mathrm{v}+\alpha+1)}\ \mathrm{J}_{\mathrm{v}+\alpha+1}(\mathrm{x}) \tag{11}$$

Theorem 1. [11] Let A be an integer sequence and B its Binomial transform. Then A *and* B *have*the same Hankel transform.
Proof. Let $A = \{a_1, a_2, a_3 \ldots .\}$ $B = \{b_1, b_2, b_3 \ldots .\}$ and and define H^* to be the matrix$H^* = RHC$, where the elements of $R, H,$ *and* $C$are given by

$$r_{i,j} = \begin{cases} 0, if\ i < j \\ \binom{i-1}{j-1}, if\ i \geq j \end{cases} \quad , \quad h_{k,m} = a_{k+m-1}$$

and

$$c_{i,j} = \begin{cases} 0, if\ i < j \\ \binom{j-1}{i-1}, if\ i \geq j \end{cases}$$

And $\binom{i}{j}$ denotes the usual binomial coefficient. Then the elements of H^* are

$$h^*_{i,j} = \sum_{k=1}^{i} \sum_{m=1}^{j} \binom{i-1}{k-1} a_{k+m-1} \binom{j-1}{m-1}$$

which, by making slight changes of variables, gives

$$h^*_{i,j} = \sum_{k=0}^{i-1} \sum_{m=0}^{j-1} \binom{i-1}{k} \binom{j-1}{m} a_{k+m-1}$$

By the well-known Vandermonde convolution formula [4] and another slight change of variable, this reduces to

$$h^*_{i,j} = \sum_{s=1}^{i+j-1} \binom{i+j-2}{s-1} a_s$$

which, by the definition of the Binomial transform (see [5]), is b_{i+j-1}, thus showing that H^* is the Hankel matrix of sequence B. Thus the terms of the Hankel transforms of the sequences A and B are $det(H_n)$ and $\det(R_n H_n C_n)$ respectively, where R_n, H_n, and C_n are the upper-left submatrices of order (12) n of H, R and CH, respectively. But R_n and C_n are both triangular with all 1's on the main diagonal, thus $det(R_n)$ and $det(C_n)$ are both 1, and therefore

$$det(H_n) = det(R_n H_n C_n)$$

, completing the proof.

Theorem 2. Let A be an integer sequence and B its Invert transform. Then A and B have the same

Hankel transform.

Proof Let $A = \{a_1, a_2, a_3 \ldots.\}$ and $B = \{b_1, b_2, b_3 \ldots.\}$and define H^* to be the matrix$H^* = RHC$, where the elements of $R, H, and\ C$ are given by

$$r_{i,k} = \begin{cases} 0, if\ k > i \\ b_{i-k}, if\ k \le i \end{cases} \quad , \quad h_{k,m} = a_{k+m-1} \quad \text{and}$$

$$c_{m,j} = \begin{cases} 0, if\ j < m \\ b_{i-k}, if\ j \ge m \end{cases}$$

where b_0 is defined to be 1. Then the $(i, j-1)$ −element of H^* given by

$$h^*{}_{i,j-1} = \sum_{k=1}^{i} \sum_{m=1}^{j-1} b_{i-k} a_{k+m-1} b_{j-m-1}$$

$$= \sum_{k=2}^{i} \sum_{m=1}^{j-1} b_{i-k} a_{k+m-1} b_{j-m-1} + b_{i-1} \sum_{m=1}^{j-1} a_m b_{j-m-1}$$

$$= \sum_{k=1}^{i-1} \sum_{m=1}^{j-1} b_{i-k}\ a_{k+m}\ b_{j-m-1} + b_{i-1} \left[\sum_{m=1}^{j-1} a_m b_{j-m-1} + a_{j-1} \right]$$

$$= \sum_{k=1}^{i-1} \sum_{m=2}^{j} b_{i-1-k}\ a_{k+m-1}\ b_{j-m} + b_{i-1} b_{j-1}$$

$$= \sum_{k=1}^{i-1} \sum_{m=1}^{j-1} b_{i-1-k}\ a_{k+m-1}\ b_{j-m} + b_{i-1} \sum_{m=1}^{j-1} b_{i-1-k}\ a_k + b_{i-1} b_{j-1}$$

$$= h^*_{i-1,j}$$

showing that elements of H* are constant along anti-diagonals. But, clearly,

$$h^*_{1,j} = \sum_{k=1}^{1} \sum_{m-1}^{j} b_{1-k}\, a_{k+m-1}\, b_{j-m}$$

$$= b_0 \sum_{m=1}^{j} a_m\, b_{j-m} = b_j$$

the last step following from the definition of the Invert transform ,which shows that $h^*_{1,j} = b_{i+j-1}$ or, in other words, that H* is the Hankel matrix of B. Since L and R are triangular with diagonals consisting of all 1's, this shows that the Hankel determinants of B are the same as those for A, and thu$A$ *and* Bhave the same Hankel transform. (13)

Calculation of The Hankel Transform Using Preliminary Wavelet

Transform

The Hankel transform is a very useful instrument in a wide range of physical problems which have an axial symmetry (POSTNIKOV, ABOUT CALCULATION OF THE HANKEL TRANSFORM USING PRELIMINARY WAVELET TRANSFORM, 2003). The influence of the Laplacian on a function in a cylindrical coordinates is equal to the product of the squared parameter of the transformation and the transform of the function(14)

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) f(r) \leftrightarrow -p^2 F_0(p)$$

(16)

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\ \frac{d}{dr} - \frac{1}{r^2}\right) f(r) \leftrightarrow -p^2 F_1(p)$$

The Hankel transforms of the null $(n = 0)$ and the first $(n = 1)$ kind are represented as

$$F_n(p) = \int_0^{\infty} f(r) J_n(pr)\, r dr$$

$$f_n(p) = \int_0^{\infty} F(p) J_n(pr)\, p dp \qquad (17)$$

Besides, those integrals like (2) are connected with the problems of geophysics and cosmology, for example, [6, 8].However, practical calculation of direct and inverse Hankel transform is connected with two problems(15). The first problem is based on the fact that not every transform in the real physical situation has analytical expression for result of inverse Hankel transform. The second one is the determination of functions as a set of their values for numerical calculations. Large bibliography on those issues can be found , The classical trapezoidal rule, Cotes rule, and other rules connected with the replacement of integrand by sequence of polynoms have high accuracy if integrand is a smooth function. (16)

But $f(r)J_n(pr)r$ $(or\ F_p(p)J_n(pr)p)$ is a quick oscillating function if r (or p) is large. There are two general methods of the effective calculation in this area. The first is the fast Hankel transform [7]. The specification of that method is transforming the function to the logarithmical space and fast Fourier transform in that space. This method needs a smoothing of the function in logs pace. The second method is based on the separation of the integrand into product of slowly varying component and a rapidly oscillating Bessel function [2]. But it needs the smoothness of the slow component for its approximation by low-order polynoms.

The goal of this paper is to apply wavelet transform with Haar bases to (2). The both direct and inverse transforms (2) are symmetric. Consider only one of them, for example, direct transform. Denote $f(r)r\ as\ g(r)$. Then, the Hankel transform is (17)

$$F_{0,1}(pr) = \int_0^\infty g(r) J_{0,1}(pr)\, rdr \tag{18}$$

The expansion $g(r) \in L^2(R)$ into wavelet series with the Haar bases is (see [3])

$$g(r) = \sum_{k=0}^{\infty} c_{0k}\, \varphi_k(r) + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} d_{jk} \Psi_{jk}(r) \tag{19}$$

$$\Psi_{0k}^H(r) = \varphi^H(r-k) \qquad \Psi_{jk}^H(r) = 2^{j/2}\varphi^H(2^{j/2} r - k)$$

(3.9.4.5)

$$\varphi^H = \begin{cases} 1 \ , & t \in (0,1) \\ 0 \ , & t \notin (0,1) \end{cases} \qquad \varphi^H(t) = \begin{cases} 1 \ , & t \in (0,\frac{1}{2}) \\ -1 \ , & t \in (\frac{1}{2},0) \\ 0 \ , & t \notin (0,1) \end{cases} \qquad (3.9.4.6)$$

After substituting

(4) into (3) , one has

$$F_{0,1}(p) = \sum_{k=0}^{\infty} c_{0k} \int_0^\infty \varphi_k(r) J_{0,1}(pr)\, rdr$$
$$+ \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} d_{jk} \int_0^\infty \varphi_{jk}(r) J_{0,1}(pr)\, rdr \tag{21}$$

Making use of integrals of [1] , we have , as result ,

$$\begin{aligned} F_1(p) = \frac{1}{p}\Bigg\{ & \sum_{k=0}^{\infty} c_{0k}[(k+1)J_0(p(k+1)) - kJ_0(pk) \\ & + \frac{\pi}{2}[(k+1)D(p(k+1)) - kD(pk)]] \\ & + \sum_{j=0}^{\infty}\sum_{k\in z} d_{jk}[2\left(k+\frac{1}{2}\right) J_0\left(p\left(k+\frac{1}{2}\right)\right) - (k+1)J_0(p(k+1)) \\ & - kJ_0(pk) \\ & - \frac{\pi}{2}\left[2\left(k+\frac{1}{2}\right) D\left(p\left(k+\frac{1}{2}\right)\right) - (k+1)D(p(k+1)) - kD(pk)\right]]\Bigg\} \end{aligned} \tag{22}$$

$$\begin{aligned} F_1(p) = \frac{1}{p}\Bigg\{ & \sum_{k=0}^{\infty} c_{0k}[J_0(pk) - J_0(p(k+1))] \\ & + \sum_{j=0}^{\infty}\sum_{k\in z} d_{jk}[2J_0\left(p\left(k+\frac{1}{2}\right)2^{-j}\right) - J_0(p(k+1)2^{-j})]\Bigg\} \end{aligned} \tag{23}$$

where $D(\xi) = H_0(\xi)J_1(\xi) - H_1(\xi)J_0(\xi)$ and $H_{0,1}$ is a Struve function the null and the first kind.

The most sufficient result is that (8) and (9) are exact. They can be use in any analytical expressions. Especially it is useful for Hankel transfor of the first kind because (9) contains only a combination of Bessel functions, and one can use their properties such as orthogonally, [18] known locality of the zeros, and extremums. The coefficients c_{0k} means average value of $g(r)$ at the range ,$k, k + 1$- is

$$c_{0k} = \int_{k}^{k+1} g(r)dr \qquad (24)$$

The detail coefficients are

$$d_{jk} = 2^{j/2}\left\{\int_{2^{-jk}}^{2^{-j}(k+\frac{1}{2})} g(r)dr - \int_{2^{-j}(k+\frac{1}{2})}^{2^{-j}(k+1)} g(r)dr\right\} \qquad (25)$$

Formulas (8) and (9) allow us to get a full analytical solution if the integrals above have close form solution. In the opposite case, the solution must be numerical but this method provides an effective algorithm for that. It is obvious that d_{jk}decrease very quickly if $g(r)$ is a smooth function. One can practically use$d_{jk} > \varepsilon$, where ε is small. The largest detail coefficients are concentrated around steps, sharp vertices, and discontinues of $g(r)$; and one can appropriate that they are equal to zero in other areas[19].
Consider, for example, a function with known analytical Hankel transform

$$\int_0^{\infty} e^{-a^2r^2} rJ_1(pr)rdr = \frac{p}{4a^4} e^{-p^2/4a^2} \qquad (26)$$

The approximation and detail coefficients may be calculated analytically in a closed form

$$c_{0k} = \left.\frac{\sqrt{\pi}\,\mathrm{erf}(r) - 2are^{-a^2r^2}}{4a^3}\right|_{k}^{(k+1)}$$

$-j$

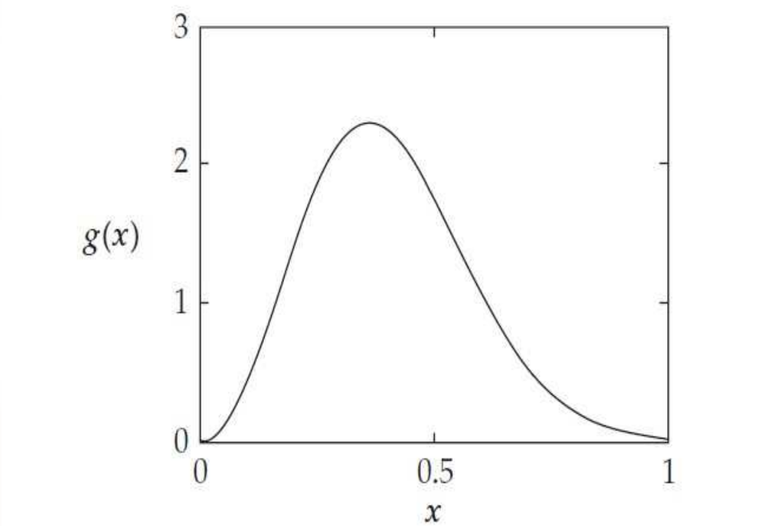
$$d_{jk} = 2^{j/2} \frac{\sqrt{\pi}\,\mathrm{erf}(r) - 2are^{-a^2r^2}}{4a^3}\Bigg|_{2^{-jk}}^{2\quad(k+1/2)}$$

(27)

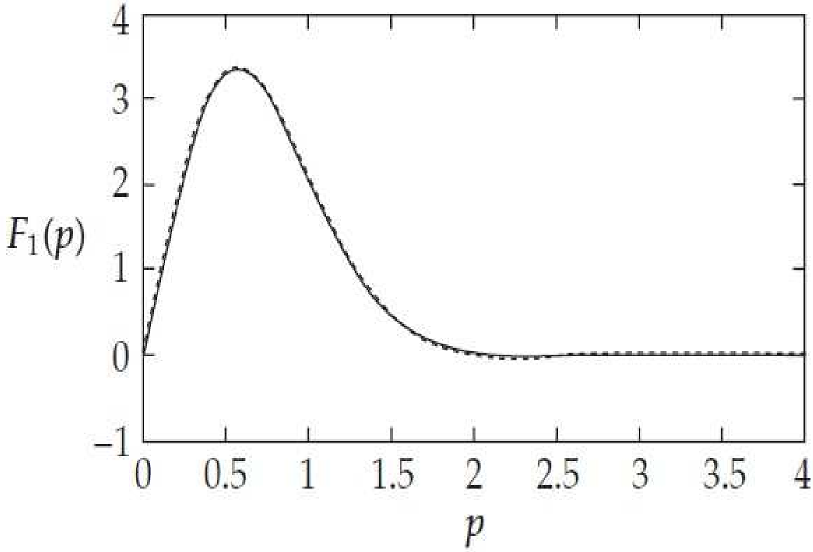
$$-2^{j/2} \frac{\sqrt{\pi}\,\mathrm{erf}(r) - 2are^{-a^2r^2}}{4a^3}\Bigg|_{2^{-j}(k+1/2)}^{2^{-j}(k+1)}$$

Thus (23), with the coefficients (27), is the exact representation of the Hankel transform. Consider the approximate solution. Suppose that the function (26) is known only in the *segment* $[0, h]$. Then there is the series, instead of (19),

$$g(r) = c_{0k}\varphi_0(r) + \sum_{j=0}^{J}\sum_{k=0}^{2^J-1} d_{jk}\Psi_{jk}(r) \quad ()$$

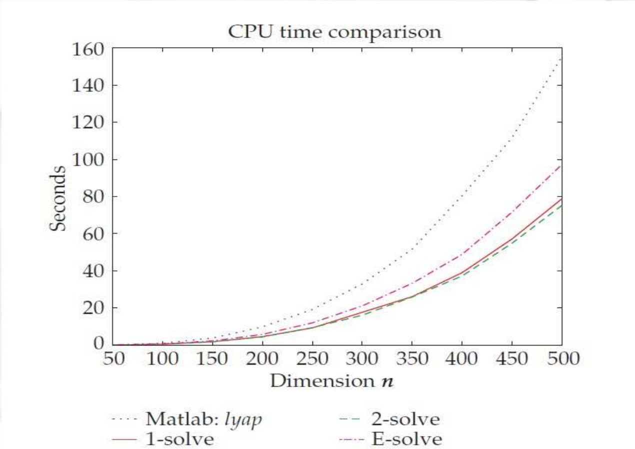


(a)

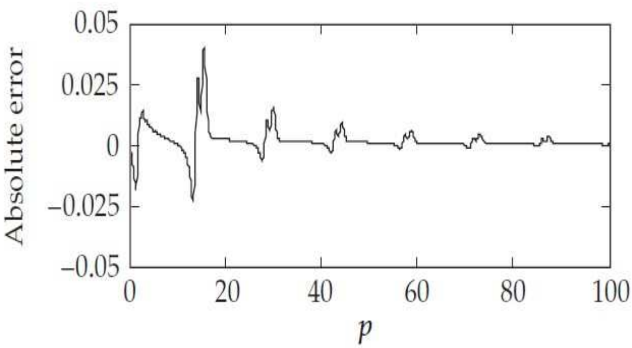


(b)

If $J \to$, then (14) is exact for this truncated function. In practice, one uses only small $J, up\ to\ 3-4$. For example, we can see the original function (12) (the replacement $r\ to\ x = r/h$ is used) and the transform in Figure 1. One can see that the exact transform (solid line) and the transform at level $J = 3$ (dotted line) coincide in this figure. The absolute errors between the exact transform and the approximate transform at the levels$J = 2$ (solid line), $J = 3$(dashed line), and $J = 4$(dotted line) are represented in Figure 2a. It is oblivious that the error is small in comparison with the values of the $F_1(p)$. The absolute error at the level $J = 3$ in a wide range of p is plotted in Figure 2a. One can see that this error has quasi periodic oscillations because the function is truncated. But they



(a)



(b)

decrease with the growth of p $(and\ J\ )$ when oscillations in classical fast Hankel transform [6] increase.

Heat Conduction

Heat is supplied at a constant rate Q per unit area and per unit time throu a circular disc of radius $in\ the\ plane\ z = 0$, to the semi-infinite spa $z > 0$. The thermal conductivity of the space is k . The plane $z = 0$ outsi the disc is insulated . The mathematical model of this problem is ve similar to that of section 5.1 . The temperature is denoted by $v(r, z)$. V have again the Laplace Equation (3.26) in polar coordinates , but t boundary condition are now [20]

$$-k\frac{\partial v(r,z)}{\partial z} = Q\ , \quad r < a\ , z = 0$$
$$= 0\ , \quad r > a\ , z = 0 \qquad (28)$$

The Hankel transform of the differential equation is again

$$\frac{\partial^2 V}{\partial z^2}(s,z) - s^2 V(s,z) = 0 \qquad (29)$$

We can now transform also the boundary condition , using formula: (21)

$$-k\frac{\partial v}{\partial z}(s,0) = Qa\ J_1(as)/s\ , \qquad (30)$$

The solution of (3.9.13) must remain finite as z tends to infinity . We have

$$V(s,z) = A(s)e^{-sz}$$

Using the condition (3.9.15) we can determine

$$A(s) = Qa\ \ J_1(as)/(Ks^2)$$

Consequently , the temperature is given by

$$v(r,z) = \frac{Qa}{K}\int_0^\infty e^{-sz}\ J_1(as)J_0(rs)s^{-1}ds \qquad (31)$$

Conclusions and Future outlook :

We found Hankel transform is very useful for solving solution of ordinary differential equation (ODEs) and partial differential equation (PDEs) using Hankel transform methods and compare the new solutions with other methods, apply the Hankel transform method in many different areas of physics and engineering . In this paper we used some mathematics program (Matlab) to given analytical solution for rationalized Haar wavelets , and it can used for solve anther problem by change in the final equation.

This method very efficient, easy and not wide use in solving some problem in ordinary differential equations (ODEs).and Partial differential equations (PDEs) compared with the other methods .

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