Topological vector space

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Abstract :

In this paper, we introduce the concept of weak linear topology on a topological vector space as a generalization of usual weak topology space .In the case, we show that the application of vector space topology .

المستخلص:-

هذه الورقة العلمية تختص بدراسة الفضاء التبولجي المتجه والذي يصنف كفضاء ضعيف باستخدام فضاء باناخ.

Key words : Topological space, Separation theorems , Spaces of smooth functions, Banach algebras.

1-Introduction:-

The general topology which needs some mathematical concepts **of vector Space.**Suppose that V is a set upon which we have defined two operations ***addition***, which combines two elements of V and is denoted by "+", and . ***multiplication***, which combines a complex number with an element of V and is denoted by just aposition.

One of the most important concepts is the mathematical logic and how its use facilities in proving different problems. Some concepts of the mathematical analyses like differentiation and integration must be studied before.

Definition 1-1- Topological space. A topological space is a set S with a collection τ of subsets (called the open sets) that contains both S and ϕ , and is closed under arbitrary union and finite intersections. A topological space is the most basic concept of a set endowed with a notion of neighborhood.

Definition 1-2- Open neighborhood. In a topological space (S, τ), a neighborhood of a point x is an open set that contains x. We will denote the collection of all the neighborhoods of x by

$$N_x = \{U \in \tau \mid x \in U\}.$$

Topological spaces are classified according to certain additional properties that they may satisfy. A property satisfied by many topological spaces concerns the separation between points:

Definition 1-3- Hausdorff space. A topological space is a Hausdorff space if distinct points have distinct neighborhoods, i.e., $\forall x \neq y$ there are $U_x \in N_x$ *and* $U_y \in N_y$ such that

The Hausdorff property is required, for example, for limits to be unique.

Definition1-4- Base. In a topological space (S,τ), a collection $\tau' \subset \tau$ of open sets is a base for τ if every open set is a union of members of τ'.

Bases are useful because many properties of topologies can be reduced to statements about a base generating that topology.

Example 1-5-The open balls

$$B(x.a) = \{ y \in S \mid d(y.x) < a\}$$

form a base for a topology in a metric space.

Definition 1-6- Local base.Let (S, τ) be a topological space and let $x \in S$. . collection $\gamma_x \subset N_x$ is called a local base at x if every neighborhood of x co an element of γ_x.It is easy to see that the union of all local bases is a base fo topology. Indeed, let $A \in t$. Let $x \in A$. Since A is a neighborhood of x, the the definition of the local base, there exists a $U_x \in \gamma_x$ such that $U_x \subset A$. Clearly,

$$A = \bigcup_{x \in A} U_x$$

which proves that $\cup_{x \in S} \gamma_x$ is a base for τ.

Definition 1-7- Closure. Let (S, τ) be a topological space and let E ⊂ S. The closure of E, denoted $\bar{E}$, is the set of all points all of whose neighborhoods intersect E:

$$\bar{E} = \{x \in S \mid \forall U \in \mathcal{N}_x, U \cap E \neq \phi\}.$$

Definition 1-8- Induced topology. Let (S, τ) be a topological space and let E ⊂ S. Then τ ∩E is called the induced topology on E.

Definition 1-9- Limit. A sequence (x_n) in a Hausdorff space (S, τ) converges to a limit $x \in S$ if every neighborhood of x contains all but finitely many points of the sequence.So that two notions: that of a vector space and that of a topological space.

Definition 1-10-Topological vector space. Let V be a vector space endowed with a topology t. The pair (V , τ) is called a topological vector space if

1. For every point x ∈ V , the singleton {x} is a closed set (namely, {x}c ∈ t).

2. The vector space operations are continuous with respect to t.

Comment 1-11- The first condition is not required in all texts.

The second condition means that the mappings:

$$V \times V \to V \qquad (x, y) x + y$$

$$F \times V \to V \qquad (a, x) ax$$

are continuous. That is, for every $x, y \in V$,

$$(\forall V \in \mathcal{N}_{x+y})(\exists V_x \in \mathcal{N}_x, V_y \in \mathcal{N}_y) : (V_x + V_y \subset V),$$

and for every $x \in V$ and $a \in F$,

$$(\forall V \in \mathcal{N}_{ax})(\exists V_x \in \mathcal{N}_x, V_a \in \mathcal{N}_a) : (V_a \cdot V_x \subset V).$$

Having defined a topological vector space, we proceed to define notions concerning subsets:

Definition 1-12-Bounded set. Let χ be a topological vector space. A subset

$E \subset \chi$ is said to be bounded if

$$(\forall \in N_0)(\exists s \in \mathbb{R}) : (\forall t > s)(E \subset tV).$$

That is, every neighborhood of zero contains E after being blown up sufficiently.cide with the metric notion of boundedness.

Definition 1-13- With every $a \in \chi$ we associate a **translation operator,** $Ta : \chi \to \chi$, defined by

$$T_a x = x + a,$$

and with every $0 \neq a \in F$ we associate a **multiplication operator,** $Ma : \chi \to \chi$, defined by

$$M_a x = ax.$$

Proposition 1-14- Both T_a and M_a are homeomorphisms of χ onto χ .

Proof. Both T_a and M_a are bijections by the vector space axioms. Their inverses are T−a and M1a. All are continuous by the very definition of a topological vector space.

Corollary 1-15 Every open set is translationally invariant: $E \subset \chi$ is open if and only if $a + E$ is open for every $a \in \chi$. In particular,

$$N_a = a + N_0.$$

Hence the topology is fully determined by the neighborhoods of the origin. this a classification of various types of topological vector spaces:

Definition 1-16 Let (χ, τ) be a topological vector space.

1. χ is **locally convex** if there exists a local base at 0 whose members are convex.
2. χ is **locally bounded** if 0 has a bounded neighborhood.
3. χ is **locally compact** if 0 has a neighborhood whose closure is compact
4. χ is **metrizable** if it is compatible with some metric d (i.e., τ is generated by the open balls $B(x,a) = \{y \in \chi\ d(y,x) < a\}$).
5. χ is an **F-space** if its topology is induced by a complete translationally invariant metric. Every Banach space is an F-space. An F-space is a

Banach space if in addition $d(\alpha x, 0) = |\alpha| d(x, 0)$.

6. χ is a **Frechet space** if it is a locally convex F-space.
7. χ is **normable** if it can be endowed with a norm whose induced metric is compatible with τ.
8. χ has the **Heine-Borel property** if every closed and bounded set is compact.

Hahn-Banach properties. Weak topologies, which we will investigate later are always locally convex. We will prove that the only topological vector spaces that are locally compact are finite dimensional. We will prove that a topological vector space is metrizable if it has a countable local base at the origin, which in turn, is guaranteed if the space is locally bounded. We will prove that a topological vector space is normable if and only if it is both locally convex and locally bounded.

2 - Separation theorems

A topological vector space can be quite abstract. All we know is that there is a vector space structure and a topology that is compatible with it. We have to start make our way from these very elementary concepts.

Lemma 2-1 Let χ be a topological vector space. $W \in \mathcal{N}_0\ \exists U \in \mathcal{N}_0^{sym}$ such that $U + U \subset W$.

Proof Since 0+0 = 0 and addition is continuous, The here exist neighborhoods $V_1, V_2 \in \mathcal{N}_0$ such that

$$V_1 + V_2 \subset W$$

Set

$$U = V_1 \cap (-V_1) \cap V_2 \cap (-V_2)$$

U is symmetric; it is an intersection of four open sets that contain zero, hence it is a non-empty neighborhood of zero. Since U $\subset V_1$ and U $\subset V_2$ it follows that

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Lemma 2-2 Let χ be a topological vector space.

$$\forall W \in \mathcal{N}_0\ \exists V \in \mathcal{N}_0^{sym}\ \textit{such that}\ V + V + V + V \subset W.$$

Theorem 2 -3 Let χ be a topological vector space. Let $K, C \subset \chi$ satisfy:

K is compact, C is closed and $K \cap C = \phi$.

Then there exists a $V \in \mathcal{N}_0$ such that

$$(K+V) \cap (C+V) = \emptyset .$$

In other words, there exist disjoint open sets that contain K and C.

Comment 2-4 Thus, a topological vector space is regular (a topological space is regular if separates points from closed sets that do not include that point).

Example 2-5 (The open balls)

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$$A = \bigcup_{x \in A} U_x$$

which proves that $\cup_{x \in S} \gamma_x$ is a base for τ.

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$$(\forall V \in N_{x+y})(\exists V_x \in N_x, V_y \in N_y) : (V_x + V_y \subset V),$$

and for every $x \in V$ and $a \in F$,

$$(\forall V \in N_{ax})(\exists V_x \in N_x, V_a \in N_a) : (V_a \cdot V_x \subset V).$$

Having defined a topological vector space, we proceed to define notions concerning subsets:

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Definition 2-13 Balanced set. Let χ be a topological vector space. A subset $E \subset \chi$ is said to be balanced if

$$\forall a \in F, \qquad |a| \leq 1 \qquad aE \subset E.$$

We denote the set of balanced neighborhoods of zero by $\mathcal{N}_0^{bal}$ bal

The notion of being balanced is purely algebraic. When we talk about balanced neighborhoods we connect the algebraic concept to the topological concept. Note that in $\mathbb{C}$ the only balanced sets are discs and the whole of $\mathbb{C}$.

Definition 2-14 Symmetric set. Let χ be a topological vector space. A

subset $E \subset \chi$ is said to be symmetric if

$$x \in E \text{ implies } (-x) \in E,$$

namely $(-E) = E$. We denote the set of symmetric neighborhoods of x by $\mathcal{N}_x^{sym}$

Symmetry is also an algebraic concept. Every balanced set is symmetric

.

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Proof. Both T_a and M_a are bijections by the vector space axioms. Their inverses are T−a and M1a. All are continuous by the very definition of a topological vector space.

Corollary 2-17 Every open set is translationally invariant: $E \subset \chi$ is open if andonly if $a + E$ is open for every $a \in \chi$. In particular,

$$\mathcal{N}_a = a + \mathcal{N}_0.$$

Hence the topology is fully determined by the neighborhoods of the origin.

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3- Separation theorems

A topological vector space can be quite abstract. All we know is that there is a vector space structure and a topology that is compatible.

Lemma 3-1 Let χ be a topological vector space.

$\forall W \in \mathcal{N}_0\ \exists U \in \mathcal{N}_0^{sym}$ such that $U + U \subset W$.

Proof. Since 0+0 = 0 and addition is continuous, there exist neighborhoods $V_1, V_2 \in \mathcal{N}_0$ such that

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Set

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Theorem 3 -3 Let χ be a topological vector space. Let $K, C \subset \chi$ satisfy: K is compact, C is closed and $K \cap C = \phi$.

Then there exists a $V \in \mathcal{N}_0$ such that

$$(K + V) \cap (C + V) = \emptyset .$$

The are there exist disjoint open sets that contain K and C.

Comment 3-4 The a topological vector space is regular (a topological space is regular if separates points from closed sets that do not include that point .

Proof - Let $x \in K$. Since C^c is an open neighborhood of x, it follows from the above lemma that there exists a $V_x \in \mathcal{N}_0^{sym}$ such that

$$x + V_x + V_x + V_x \in C^c$$

i.e

$$(x + V_x + V_x + V_x) \cap C = \emptyset .$$

Since V_x is symmetric,

$$(x + V_x + V_x + V_x) \cap (C + V_x) = \emptyset .$$

For every $x \in K$ corresponds such a V_x. Since K is compact, there exists a finite collection of $(x_i, V_{xi})_{i=1}^{n}$ such that

$$K \subset \cup_{i=1}^{n} (x_i + V_{xi})$$

$V = V_{x1} \cap \cdots \cap V_{xn}$.

Then, for every i,

$$(x + V_{xi} \; + V_{xi}) \text{ does not intersect } (C + V_{xi})$$

and a for interior,

$(x + V_{xi} \; + V_{xi})$ does not intersect $(C + V)$.

Taking the union over i:

$K \; + V \; \subset \bigcup_{i=1}^{n}(x_i + V_{xi} + V)$does not intersect $(C + V)$.

Comment 3-5 Since $K + V$ and $C + V$ are both open and mutually disjoint, it follows that also

$$\overline{K + V} \text{ does not intersect } C + V.$$

Indeed, if $x \; \in C + V$ then there exists U $\in \mathcal{N}_x$ such that

$$U \; \subset C + V$$

But since x $\in \overline{K \; + V}$, every neighborhood of x, and U in particular, intersects $K \; + V$, i.e., where exists a $y \; \in U$ satisfying

$$y \; \in \; (K \; + V) \cap (C + V),$$

which is a contradiction.

Corollary 3-6 Let $\mathcal{B}$ be a local base at zero for a topological vector space χ .

Then every member in $\mathcal{B}$ contains the closure of some other member in $\mathcal{B}$. That is,

$$\forall U \; \in \mathcal{B} \; \exists W \; \in \mathcal{B} \quad \text{such that} \quad \overline{W} \; \subset U.$$

Proof. Let $U \; \in \mathcal{B}$. Let $K \; = \; \{0\}$ (compact) and $C \; = U^c$ (closed). By Theorem 3.5 there exists a $V \; \in \mathcal{N}_0^{sym}$, such that

$$V \; \cap (U^c \; + V) \; = \emptyset \, .$$

It follows that

$$V \; \subset \; (U^c \; + V)^c \; \subset U.$$

By the definition of a local base there exists a neighborhood $W \; \in \mathcal{B}$ such that

$$W \subset V \subset (U^c + V)^c \subset U.$$

$(U^c + V)^c$ is closed,

$$W \subset (U^c + V)^c \subset U$$

llary 3-7 Every topological vector space χ is Hausdorff.

osition 3-8 Let χ be a topological vector space.

For $A \subset \chi$,

$$\bar{A} = \bigcap_{V \in \mathcal{N}_0} (A + V)$$

at is, the closure of a set is the intersection of all the open neighbo
; of that set.

$\bar{A} + \bar{B} \subset \overline{A + B}$.

If $\mathcal{Y} \subset \chi$ is a linear subspace, then so is $\bar{\mathcal{Y}}$.

f C is convex so is $\bar{C}$.

f C is convex so is C^o.

For every $B \subset \chi$: if B is balanced so is $\bar{B}$.

For every $B \subset \chi$: if B is balanced and $0 \in B^o$ then B^o is balanced.

If $E \subset X$ is bounded then so is $\bar{E}$.

f .

$x \in \bar{A}$. By definition, for every $V \in \mathcal{N}_0, x + V$ intersects A, of
$A - V$.Thus,

$$x \in \bigcap_{V \in \mathcal{N}_0} (A - V) = \bigcap_{V \in \mathcal{N}_0} (A + V)$$

Conversely, suppose that $x \notin \bar{A}$. Then, there exists a $V \in N_0$ such that $x + V$ does not intersect A, i.e., $x \notin A - V$, hence

$$x \notin \bigcap_{V \in N_0} (A + V)$$

conclusion:

many properties of topologies can be reduced to statements about a base generating that topology. Then the topological vector spaces that are locally compact are finite dimensional. And a topological vector space is metrizable if it has a countable local base at the origin, which in turn, is locally bounded.

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