

# A system of Integral Equations Via Measure of Non-Compactness Existence of Solutions in Banach` Spaces

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## Abstract:

This study aims to show important Solutions, possible applications, in the system of Integral Equations Via Measure of Non-Compactness and Existence of Solutions in Banach` Spaces. and present some results on the existence of coupled fixed points for a class of condensing operators in Banach spaces. Moreover, as an application, we study the problem of existence of solutions for system of nonlinear integral equations uniformly, then the system of equations has at least one solution in the space.

Keywords: space /Measure of non-compactness/Fixed point theorem/Integral

المستخلص

تهدف هذه الدراسة إلى إظهار الحلول المهمة والتطبيقات الممكنة في نظام المعادلات التكاملية عبر قياس عدم الانضغاط ووجود الحلول في فضاءات باناخ. وقدم بعض النتائج حول وجود نقاط ثابتة مقترنة لفتة من عوامل التكتيف في فضاءات باناخ. علاوة على ذلك، كتطبيق، ندرس مشكلة وجود حلول لنظام المعادلات التكاملية غير الخطية بشكل موحد، ثم نظام المعادلات التي لديها حل واحد على الأقل في الفضاء. الكلمات المفتاحية: الفضاء / قياس عدم التوافق / نظرية النقطة الثابتة / التكامل

## 1.0 Introduction:

Recently, there have been several successful efforts to apply the concept of measure of no compactness in the study of the existence and behavior of solutions of nonlinear differential and integral equations ([21, 11, 12, 13, 15, 5, 16, 23,24, 25, 26, 17, 27, 29]). In , we present and prove some new existence theorems for solutions of systems of nonlinear equations which are formulated in terms of condensing operators in Banach spaces (i.e. mappings under which the image of any set is in a certain sense more compact than the set itself <sup>(1)</sup>). Moreover, as an application, we study the problem of existence of solutions for the following system of nonlinear integral equation

$$\begin{cases} x(t) = f_1(t, x(\zeta_1(t)), y(\zeta_1(t)), \int_0^{\beta_1(t)} g_1(t, s, x(\eta_1(s)), y(\eta_1(s))) ds) \\ y(t) = f_2(t, x(\zeta_2(t)), y(\zeta_2(t)), \int_0^{\beta_2(t)} g_2(t, s, x(\eta_2(s)), y(\eta_2(s))) ds) \end{cases} \quad (1)$$

where  $f_i, g_i, \zeta_i, \eta_i$  and  $\beta_i$  satisfy certain condition

### 1.1 Objective of the Study

This study aims to fulfill the following objective:

show important Solutions, possible applications, in the system of Integral Equations Via Measure of Non-Compactness and Existence of Solutions in Banach Spaces.

### 1.2 The importance of the study:

The importance of this study is the:

1. prove some existence theorems for systems of equations involving condensing operators using the Darbo fixed point theorem.
2. investigate the problem of existence of solutions for the system of nonlinear integral equation.

### 2.1 Preliminaries:

We some basic notations, definitions and auxiliary devoted to state and prove some existence theorems for systems of equations involving condensing operators using the Darbo fixed point theorem. Finally, using the obtained results, we investigate the problem of existence of solutions for the system of nonlinear integral equation (1).

The first measure of non-compactness was defined by Kuratowski <sup>(2)</sup>. In a metric space  $X$  and for a bounded subset  $S$  of  $X$  the Kuratowski measure of noncompactness is defined as

$$\alpha(S) := \inf \left\{ \delta > 0 \mid S = \bigcup_{i=1}^n S_i, \text{ for some } S_i, \text{ with } \text{diam}(S_i) \leq \delta \text{ for } 1 \leq i \leq n \leq \infty \right\} \quad (2)$$

Here  $\text{diam}(T)$  denotes the diameter of a set  $T \subset X$ , i.e.,

$$\text{diam}(T) := \sup \{d(x, y) \mid x, y \in T\}.$$

Another important measure of non-compactness is the so-called Hausdorff (or ball) measure of non-compactness defined as

$$\chi(X) = \inf \{ \varepsilon : X \text{ has a finite } \varepsilon - \text{net in } E \}.$$

Since a ball of radius  $r$  has diameter at most  $2r$ , then the measures  $\chi$  and  $\alpha$  are equivalent i.e., for any bounded subset  $X$  of  $E$  the following estimate holds<sup>(3)</sup>

$$\chi(X) \leq \alpha(X) \leq 2\chi(X).$$

The two measures  $\chi$  and  $\alpha$  share many properties [8, 5]. Here, we recall some basic facts concerning measures of non-compactness from [5], which is defined axiomatically in terms of some natural conditions. Denote by  $\mathbb{R}$  the set of real numbers and put  $\mathbb{R}_+ = [0, +\infty)$ . Let  $(E, \|\cdot\|)$  be a Banach space. The symbol  $\bar{X}$ ,  $\text{Conv}X$  will denote the closure and closed convex hull of a subset  $X$  of  $E$ , respectively. Moreover, let  $\mathfrak{M}_E$  indicate the family of all nonempty and bounded subsets of  $E$  and  $\mathfrak{K}_E$  indicate the family of all nonempty and relatively compact subsets<sup>(4)</sup>.

**Definition (1):** A mapping  $\mu: \mathfrak{M}_E \rightarrow \mathbb{R}_+$  is said to be a measure of noncompactness in  $E$  if it satisfies the following conditions<sup>(5)</sup>:

- (i) The family  $\ker \mu = \{X \in \mathfrak{M}_E: \mu(X) = 0\}$  is nonempty and  $\ker \mu \subseteq \mathfrak{K}_E$ .
- (ii)  $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$ .
- (iii)  $\mu(\bar{X}) = \mu(X)$ .
- (iv)  $\mu(\text{Conv}X) = \mu(X)$ .
- (v)  $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda\mu(X) + (1 - \lambda)\mu(Y)$  for  $\lambda \in [0, 1]$ .
- (vi) If  $\{X_n\}_n$  is a sequence of closed sets from  $\mathfrak{M}_E$  such that  $X_{n+1} \subset X_n$  for  $n = 1, 2, \dots$  and if  $\lim_{n \rightarrow \infty} \mu(X_n) = 0$  then  $X_\infty = \bigcap_{n=1}^{\infty} X_n = \emptyset$ .

Here we recall the well-known fixed point theorem of Darbo<sup>(6)</sup>.

**Definition (2)**<sup>(7)</sup>: An element  $(x, y) \in X \times X$  is called a coupled fixed point of the mapping  $F: X \times X \rightarrow X$  if  $F(x, y) = x$  and  $F(y, x) = y$ .

Note that if  $F: C \times C \rightarrow C$  is a continuous operator and we define  $F_1(x, y) = F(x, y)$  and  $F_2(x, y) = F(y, x)$  then as a result of Theorem (1) and Corollary (1) we have the main results .

### 3.1 Main results:

In this section, we state and prove some existence results for solutions of systems of equations Via Measure of Non-Compactness and Existence of Solutions in Banach` Spaces which will be used in Section

**Theorem (1)**<sup>(8)</sup>: Let  $\Omega$  be a nonempty, bounded, closed and convex subset of a space  $E$  and let  $F: \Omega \rightarrow \Omega$  be a continuous mapping such that there exists a constant  $k \in [0, 1)$  with the property

$$\mu(FX) \leq k\mu(X)$$

for any nonempty subset  $X$  of  $\Omega$ . Then  $F$  has a fixed point in the set  $\Omega$ .

The following theorem and example are basic to prove all the results of this work.

**Theorem (2)** <sup>(9)</sup>: Suppose  $\mu_1, \mu_2, \dots, \mu_n$  are measures in  $E_1, E_2, \dots, E_n$ , respectively. Moreover, assume that the function  $F: \mathbb{R}_+^n \rightarrow \mathbb{R}_+$  is convex and  $F(x_1, \dots, x_n) = 0$  if and only if  $x_i = 0$  for  $i = 1, 2, \dots, n$ . Then

$$\mu(X) = F(\mu_1(X_1), \mu_2(X_2), \dots, \mu_n(X_n))$$

defines a measure of non-compactness in  $E_1 \times E_2 \times \dots \times E_n$  where  $X_i$  denotes the natural projection of  $X$  into  $E_i$  for  $i = 1, 2, \dots, n$ .

As results from Theorem (1.3.2) we present the following example.

**Theorem (3)** <sup>(10)</sup> : Let  $C$  be a nonempty, bounded and closed subset of a Banach space  $E$  and  $\mu$  an arbitrary measure of noncompactness on  $E$ . If  $f_i: C \times C \rightarrow C$  for  $i = 1, 2$  are continuous operators and there exists a constant  $k \in [0, 1)$  such that

$$\mu(F_i(X_1 \times X_2)) \leq k \max\{\mu(X_1), \mu(X_2)\} \quad (3)$$

for any subset  $X_1, X_2$  of  $C$ , then there exist  $x^*, y^* \in X$  such that

$$\begin{cases} F_1(x^*, y^*) = x^*, \\ F_2(x^*, y^*) = y^*. \end{cases} \quad (4)$$

**Proof:** Consider the operator  $\tilde{F}: C \times C \rightarrow C \times C$  defined by

$$\tilde{F}(x, y) = (F_1(x, y), F_2(x, y)).$$

**Theorem(4)** <sup>(11)</sup>: Let  $C$  be a nonempty, bounded and closed subset of a Banach space  $E$  and  $\mu$  an arbitrary measure of noncompactness on  $E$ . If  $F_i: C^n \rightarrow C, i = 1, \dots, n$  are continuous operators for which there exists a constant  $k \in [0, 1)$  such that

$$\mu(F_i(X_1 \times \dots \times X_n)) \leq k \max\{\mu(X_1), \dots, \mu(X_n)\}$$

for any subset  $X_1, \dots, X_n$  of  $C$ . Then there exist  $x_1^*, \dots, x_n^* \in X$  such that

$$\begin{cases} F_1(x_1^*, \dots, x_n^*) = x_1^* \\ \vdots \\ F_n(x_1^*, \dots, x_n^*) = x_n^* \end{cases}$$

**Proof:** Define  $\vec{F}(x_1, \dots, x_n) = (F_1(x_1, \dots, x_n), \dots, F_n(x_1, \dots, x_n))$  and follow the proof of Theorem (1.3.5).

as an application of Theorem (1.3.5), we prove an existence result for solutions of system (25). We will work in the Banach space  $BC(\mathbb{R}_+)$  consisting of all real functions defined, bounded and continuous on  $\mathbb{R}_+$ . The space  $BC(\mathbb{R}_+)$  is furnished with the standard supremum norm i.e., the norm defined by the formula

$$\|x\| = \sup\{|x(t)|: t \geq 0\}.$$

We will use a measure of non-compactness in the space  $BC(\mathbb{R}_+)$  which is stated in ([8,9]). In order to define this measure, let us fix a nonempty bounded subset of  $X$  of  $BC(\mathbb{R}_+)$  and a positive number  $L > 0$ . For  $x \in X$  and  $\varepsilon \geq 0$  denote by  $\omega^L(x, \varepsilon)$ , the modulus of continuity of  $x$  on the interval  $[0, L]$ , i.e,

$$\omega^L(x, \varepsilon) = \sup\{|x(t) - x(s)|: t, s \in [0, L], |t - s| \geq \varepsilon\}.$$

Moreover, let us put

$$\omega^L(X, \varepsilon) = \sup\{\omega^L(x, \varepsilon): x \in X\}$$

$$\omega_0^L(X) = \lim_{\varepsilon \rightarrow 0} \omega^L(X, \varepsilon),$$

$$\omega_0(X) = \lim_{L \rightarrow \infty} \omega_0^L(X).$$

If  $t$  is a fixed number from  $\mathbb{R}_+$ , let us denote

$$X(t) = \{x(t): x \in X\}.$$

Finally, consider the function  $\mu$  defined on  $\mathfrak{M}_{BC(\mathbb{R}_+)}$  by the formula

$$\mu(X) = \omega_0(X) + \limsup_{t \rightarrow \infty} \text{diam } X(t).$$

Where

$$\text{diam } X(t) = \sup\{|x(t) - y(t)|: x, y \in X\}.$$

It can be shown (cf. [5,16]) that the function  $\mu(X)$  defines a measure of noncompactness on  $BC(\mathbb{R}_+)$  in the sense of the above accepted definition.

Now, we are ready to state and prove the main on the existence of solutions for the system of integral equations (1).

**Theorem (5)**<sup>(12)</sup>: Assume that the following conditions are satisfied:

- (i)  $\xi_i, \eta_i, \beta_i: \mathbb{R}_+ \rightarrow \mathbb{R}_+ (i = 1, 2)$  are continuous and  $\xi_i(t) \rightarrow 0$  as  $t \rightarrow \infty$  for  $i = 1, 2$ ,

(ii)  $f_i: \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  for  $i = 1, 2$  are continuous. Moreover, there exist constant  $k \in (0, 1)$  and nondecreasing continuous functions

$\Phi_i: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with  $\Phi_i(0) = 0, i = 1, 2$ , such that

$$|f_i(t, x, y, z) - f_i(t, u, v, w)| \leq k \max\{|x - u|, |y - v|\} + \Phi_i(m_i(t)|z - w|) \quad (5)$$

where  $m_i(t): \mathbb{R}_+ \rightarrow \mathbb{R}_+$  are continuous functions.

(iii) The functions  $|f_i(t, 0, 0, 0)|$  for  $i = 1, 2$  are bounded on  $\mathbb{R}_+$ , i.e.

$$M_i = \sup\{f_i(t, 0, 0, 0) : t \in \mathbb{R}_+\} < \infty. \quad (6)$$

(iv)  $g_i: \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  for  $i = 1, 2$  are continuous and there exists a positive constant  $D$  such that

$$\sup\{m_i(t) \left| \int_0^{\beta_i} g_i(t, s, x(\eta_i(s)), y(\eta_i(s))) ds \right| : t \in \mathbb{R}_+, x, y \in BC(\mathbb{R}_+), 1 \leq i \leq 2\} < D. \quad (7)$$

Moreover,

$$\lim_{n \rightarrow \infty} m_i(t) \left| \int_0^{\beta_i} [g_i(t, s, x(\eta_i(s)), y(\eta_i(s))) - g_i(t, s, u(\eta_i(s)), v(\eta_i(s)))] ds \right| = 0 \quad (8)$$

**Corollary (1)** <sup>(13)</sup>: Let  $C$  be a nonempty, bounded and closed subset of a Banach space  $E$  and an arbitrary measure of non-compactness on  $E$ . If  $F_i: C \times C \rightarrow C$  for  $i = 1, 2$  are continuous operators for which there exist nonnegative constants  $k_1, k_2$  with  $k_1 + k_2 < 1$  such that

$$\mu(F_i(X_1 \times X_2)) \leq k_1 \mu(X_1) + k_2 \mu(X_2) \quad (9)$$

for any subsets  $X_1, X_2$  of  $C$ , then there exist  $x^*, y^* \in X$  such that

$$\begin{cases} F_1(x^*, y^*) = x^*, \\ F_2(x^*, y^*) = y^*. \end{cases}$$

**Proof:** It is enough to show that (27) holds. Let  $k_1, k_2 \in C$  be given, then

$$\begin{aligned} \mu(F_i(X_1 \times X_2)) &\leq k_1 \mu(X_1) + k_2 \mu(X_2) \\ &\leq k_1 \max\{\mu(X_1), \mu(X_2)\} + k_2 \max\{\mu(X_1), \mu(X_2)\} \\ &\leq (k_1 + k_2) \max\{\mu(X_1), \mu(X_2)\}. \end{aligned}$$

Now the conclusion follows from Theorem (2).

**Corollary (2)<sup>(14)</sup>** : Let  $C$  be a nonempty, bounded and closed subset of a Banach space  $E, \mu$  an arbitrary measure of non-compactness on  $E$  and  $F: C \times C \rightarrow C$  a continuous operator. Suppose either:

(I) There exist nonnegative constants  $k_1, k_2$  with  $k_1, k_2 < 1$  such that  $\mu(F(X_1 \times X_2)) \leq k_1\mu(X_1) + k_2\mu(X_2)$ ,

or

(II) There exists a constant  $k \in [0,1)$  such that  $\mu(F(X_1 \times X_2)) \leq k \max\{\mu(X_1), \mu(X_2)\}$

for any subset  $X_1, X_2$  of  $C$ . Then  $F$  has a coupled fixed point.

**Proof:** Take  $F_1(x, y) = F_2(x, y) = F(x, y)$  in Theorem (1) and Corollary (1).

**Corollary (3)<sup>(15)</sup>**: Let  $C$  be a nonempty, bounded, closed and convex subset of a Banach space  $E$  and let  $F_i: C \times C \rightarrow E$  for  $i = 1, 2$  be operators such that

$$\|F_i(x, y) - F_i(u, v)\| \leq k \max\{\|x - u\|, \|y - v\|\}, \quad (10)$$

where  $k \in [0,1)$ . Assume that  $G_i: C \times C \rightarrow X$  are compact and continuous operators and the operators  $T_i: C \times C \rightarrow C$  defined by

$$\|T_i(x, y) - T_i(u, v)\| \leq \|F_i(x, y) - F_i(u, v)\| + \Phi(\|G_i(x, y) - G_i(u, v)\|) \quad (11)$$

for  $i = 1, 2$  where  $\Phi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a nondecreasing continuous function and  $\Phi(0) = 0$ . Then there exist  $x^*, y^* \in C$  such that

$$\begin{cases} T_1(x^*, y^*) = x^*, \\ T_2(x^*, y^*) = y^*. \end{cases}$$

**Proof:** Let  $X_1$  and  $X_2$  be arbitrary subsets of  $C$  and fixed  $1 \leq i \leq 2$ . By the definition of Kuratowski measure of non-compactness for every  $\varepsilon > 0$  there exist  $S_1, \dots, S_n$  such that  $X_1 \times X_2 \subseteq \bigcup_{k=1}^n S_k$ ,

$$\text{diam}(F_i(S_k)) < \alpha(F_i(X_1 \times X_2)) + \varepsilon$$

and

$$\text{diam}(G_i(S_k)) < \varepsilon.$$

Let us fix arbitrarily  $1 \leq k \leq n$ . Then for every  $p, q \in S_k$  we have

$$\|T_i(p) - T_i(q)\| \leq \|F_i(p) - F_i(q)\| + \Phi(\|G_i(p) - G_i(q)\|).$$

Thus, by properties of  $\Phi$  we obtain

$$\text{diam}(T_i(S_k)) \leq \text{diam}(F_i(S_k)) + \Phi(\text{diam}(G_i(S_k))),$$

$$\text{diam}(T_i(S_k)) \leq \alpha(F_i(X_1 \times X_2)) + \varepsilon + \Phi(\varepsilon)$$

and since  $\varepsilon$  was chosen arbitrarily and  $\Phi$  is a nondecreasing continuous function, so

$$\alpha(T_i(X_1 \times X_2)) \leq \alpha(F_i(X_1 \times X_2)). \tag{12}$$

Now we show that  $T_i$  satisfies (27). To do this fix arbitrary  $x, y \in X_1$  and  $u, v \in X_2$ . Then we have

$$\begin{aligned} \|F_i(x, y) - F_i(u, v)\| &\leq k \max\{\|x - u\|, \|y - v\|\} \\ &\leq k \max\{\text{diam}X_1, \text{diam}X_2\} \end{aligned}$$

so

$$\text{diam}(F_i(X_1 \times X_2)) \leq k \max\{\text{diam}X_1, \text{diam}X_2\}$$

Therefore, by definition of Kuratowski measure of non-compactness we have

$$\alpha(F_i(X_1 \times X_2)) \leq k \max\{\alpha(X_1), \alpha(X_2)\} \tag{13}$$

By (32) and (33) we deduce

$$\alpha(T_i(X_1 \times X_2)) \leq k \max\{\alpha(X_1), \alpha(X_2)\}.$$

Also, by condition (11),  $T_i$  ( $i = 1, 2$ ) are continuous operators and the application of Theorem (1) completes the proof.

In the same way as the above proof, we can extend Theorem (1) for  $n$ -dimensional systems of equations

**Example (1)** shows that  $\tilde{\mu}(X) = \max\{\mu(X_1), \mu(X_2)\}$  is a measure of non-compactness in the space  $C \times C$ , where  $X_i, i = 1, 2$  denote the natural projections of  $X$ . Now let  $X$  be any nonempty subset of  $X_i, i = 1, 2$ . Then by (ii) and (3) we obtain

$$\begin{aligned} \tilde{\mu}(\tilde{F}(X)) &\leq \tilde{\mu}(F_1(X_1 \times X_2) \times F_2(X_1 \times X_2)) \\ &= \max\{\mu(F_1(X_1 \times X_2)), \mu(F_2(X_1 \times X_2))\} \\ &\leq \max\{k \max\{\mu(X_1), \mu(X_2)\}, k \max\{\mu(X_2), \mu(X_1)\}\} \\ &\leq k\tilde{\mu}(X) \end{aligned}$$

Since  $\tilde{\mu}$  is also a measure of noncompactness, therefore all conditions of Theorem (1.3.3) are satisfied. Hence  $\tilde{F}$  has a fixed point, i.e., there exist  $x^*, y^* \in X$  such that

$$(x^*, y^*) = \tilde{F}(x^*, y^*) = (F_1(x^*, y^*), F_2(x^*, y^*)),$$

which means  $(x^*, y^*)$  solves (10).

**Example (2):**<sup>(16)</sup> Let  $\mu$  be a measure of non-compactness, considering  $F_1(x, y) = \max\{x, y\}$  and  $F_2(x, y) = x + y$  for any  $(x, y) \in \mathbb{R}_+^2$  then all the conditions of Theorem (1.3.2) are satisfied. Therefore,  $\tilde{\mu}_1 = \max\{\mu(X_1), \mu(X_2)\}$  and  $\tilde{\mu}_2 = \mu(X_1) + \mu(X_2)$  are measures of non-compactness in the space  $E \times E$  where  $X_i, i = 1, 2$  denote the natural projections of  $X$ .

we state and prove some existence results for solutions of systems of equations involving condensing operators in Banach spaces which will be used.

#### 4.1 Application:

uniformly with respect to  $x, y, u, v \in BC(\mathbb{R}_+)$  for  $i = 1, 2$ .

Then the system of equations (1) has at least one solution in the space  $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$ .

**Proof:** The proof is carried out in two steps.

**Step 1:**  $G_i: BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \rightarrow BC(\mathbb{R}_+)$  defined by

$$G_i(x, y)(t) = m_i(t) \int_0^{\beta_i(t)} g_i(t, s, x(\eta_i(s)), y(\eta_i(s))) ds \tag{14}$$

for  $i = 1, 2$  are compact and continuous operators.

Let  $1 \leq i \leq 2$  be fixed. Notice that the continuity of  $G_i(x, y)(t)$  for any  $x \in BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$  is obvious. Moreover, by (12),  $G_i$  is an operator on  $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$  into  $BC(\mathbb{R}_+)$ . Now, we show that  $G_i$  is continuous. For this, take  $x, y \in BC(\mathbb{R}_+)$  and  $\varepsilon > 0$  arbitrarily, and consider  $u, v \in BC(\mathbb{R}_+)$  with  $\|x - u\| < \varepsilon$  and  $\|v - y\| < \varepsilon$ . Then we have

$$\begin{aligned} |G_i(x, y)(t) - G_i(u, v)(t)| &\leq m_i(t) \left| \int_0^{\beta_i(t)} g_i(t, s, x(\eta_i(s)), y(\eta_i(s))) ds \right. \\ &\quad \left. - m_i(t) \int_0^{\beta_i(t)} g_i(t, s, u(\eta_i(s)), v(\eta_i(s))) ds \right| \\ &\leq m_i(t) \left| \int_0^{\beta_i(t)} [g_i(t, s, x(\eta_i(s)), y(\eta_i(s))) \right. \\ &\quad \left. - g_i(t, s, u(\eta_i(s)), v(\eta_i(s)))] ds \right|. \end{aligned}$$

Furthermore, considering condition (iv), there exists  $T > 0$  such that for  $t > T$  we have

$$|G_i(x, y)(t) - G_i(u, v)(t)| \leq \varepsilon. \tag{15}$$

Also, if  $t \in [0, T]$ , then from (39) it follows that

$$|G_i(x, y)(t) - G_i(u, v)(t)| \leq m_T \beta_T \vartheta(\varepsilon),$$

where

$$\beta_T = \sup\{\beta_i(t) : t \in [0, T], 1 \leq i \leq 2\}$$

$$m_T = \sup\{m_i(t) : t \in [0, T], 1 \leq i \leq 2\}$$

$$b = \max\{\|x\|, \|y\|\} + \varepsilon$$

$$\vartheta(\varepsilon) = \sup\{|g_i(t, s, x, y) - g_i(t, s, u, v)| : t \in [0, T], s \in [0, \beta_T], x, y, u, v \in [-b, b], |x - u| \leq \varepsilon, |y - v| \leq \varepsilon\}$$

By using the continuity of  $g_i$  on the compact set  $[0, T] \times [0, \beta_T] \times [-b, b] \times [-b, b]$ , we have  $\vartheta(\varepsilon) \rightarrow 0$ ; as  $\varepsilon \rightarrow 0$ . Thus,  $G_i$  is a continuous function from  $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$  into  $BC(\mathbb{R}_+)$ .

Now, Let  $X_1, X_2$  be two nonempty and bounded subsets of  $BC(\mathbb{R}_+)$ , and assume that  $T > 0$  and  $\varepsilon > 0$  are chosen arbitrarily. Let  $t_1, t_2 \in [0, T]$ , with  $|t_2 - t_1| \leq \varepsilon$  and  $x, y \in X$ , we obtain

$$\begin{aligned} & |G_i(x, y)(t_2) - G_i(u, v)(t_1)| \leq \\ & \leq \left| m_i(t_1) \int_0^{\beta_i(t_2)} g_i(t_2, s, x(\eta_i(s)), y(\eta_i(s))) ds \right. \\ & \left. - m_i(t_2) \int_0^{\beta_i(t_1)} g_i(t_1, s, x(\eta_i(s)), y(\eta_i(s))) ds \right| \\ & \leq m_T \left| \int_0^{\beta_i(t_2)} [g_i(t_2, s, x(\eta_i(s)), y(\eta_i(s))) \right. \\ & \quad \left. - g_i(t_1, s, x(\eta_i(s)), y(\eta_i(s)))] ds \right| \tag{16} \\ & + m_T \left| \int_{\beta_i(t_1)}^{\beta_i(t_2)} g_i(t_1, s, x(\eta_i(s)), y(\eta_i(s))) ds \right| \\ & \leq m_T \beta_T \omega_r^T(g_i, \varepsilon) + m_T U_r^T \omega^T(\beta_i, \varepsilon), \end{aligned}$$

where

$$r = \max\{\sup\{\|x\| : x \in X_1\}, \sup\{\|x\| : x \in X_2\}\}$$

$$\omega^T(\beta_i, \varepsilon) = \{|\beta_i(t_1) - \beta_i(t_2)| : t_1, t_2 \in [0, T], |t_1 - t_2| \leq \varepsilon\}$$

$$\omega_r^T(g_i, \varepsilon) = \sup\{|g_i(t_2, s, x, y) - g_i(t_1, s, x, y)| : t_1, t_2 \in [0, T], |t_2 - t_1| \leq \varepsilon\}$$

$$x, y \in [-r, r], s \in [0, \beta_T]$$

$$U_r^T = \sup\{|g_i(t, s, x, y)| : t \in [0, T], s \in [0, \beta_T], x, y \in [-r, r]\}.$$

Since  $(x, y)$  was an arbitrary element of  $X_1 \times X_2$  in (40), so we obtain

$$\omega^T(G_i(X_1 \times X_2), \varepsilon) \leq m_T \beta_T \omega_r^T(g_i, \varepsilon) + m_T U_r^T \omega^T(\beta_T, \varepsilon) \quad (17)$$

On the other hand by the uniform continuity of  $g_i$  on  $[0, T] \times [0, \beta_T] \times [-r, r] \times [-r, r]$ , we have  $\omega_r^T(g_i, \varepsilon) \rightarrow 0$ , as  $\varepsilon \rightarrow 0$  and also because of the uniform continuity of  $\beta$  on  $[0, T]$ , we derive that  $\omega_r^T(\beta, \varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Therefore we obtain

$$m_T \beta_T \omega_r^T(g_i, \varepsilon) + m_T U_r^T \omega^T(\beta_T, \varepsilon) \rightarrow 0,$$

as  $\varepsilon \rightarrow 0$  and

$$\omega_0^T(G_i(X_1 \times X_2)) = 0,$$

therefore

$$\omega_0(G_i(X_1 \times X_2)) = 0. \quad (18)$$

Finally, for arbitrary  $(x, y), (u, v) \in X_1 \times X_2$  and  $t \in \mathbb{R}_+$  we get

$$\begin{aligned} |G_i(x, y)(t) - G_i(u, v)(t)| &\leq \\ &\leq m_i(t) \left| \int_0^{\beta_i(t)} [g_i(t, s, x(\eta_i(s)), y(\eta_i(s))) \right. \\ &\quad \left. - g_i(t, s, u(\eta_i(s)), v(\eta_i(s)))] ds \right| \quad (19) \\ &\leq m_i(t) \theta_i(t), \end{aligned}$$

where

$$\begin{aligned} \theta_i(t) = \sup \left\{ \left| \int_0^{\beta_i(t)} [g_i(t, s, x(\eta_i(s)), y(\eta_i(s))) \right. \right. \\ \left. \left. - g_i(t, s, u(\eta_i(s)), v(\eta_i(s)))] ds \right| : x, y, u, v \in BC(\mathbb{R}_+) \right\} \end{aligned}$$

Since  $(x, y), (u, v)$  and  $t$  were chosen arbitrarily in (43), we conclude that

$$\text{diam} G_i(X_1 \times X_2)(t) \leq m(t) \theta(t). \quad (20)$$

Taking  $t \rightarrow \infty$  in the inequality (44), then using (iv) we deduce that

$$\limsup_{t \rightarrow \infty} \text{diam} G_i(X_1 \times X_2)(t) = 0. \quad (21)$$

Further, combining (42) and (45) we get

$$\limsup_{t \rightarrow \infty} \text{diam} G_i(X_1 \times X_2)(t) + \omega_0(G_i(X_1 \times X_2)) = 0$$

or, equivalently

$$\mu(G_i(X_1 \times X_2)) = 0.$$

Thus,  $G_i$  is a compact and continuous operator.

**Step 2:** There exists  $r_0 \in \mathbb{R}_+$  such that the operators  $T_i: \bar{B}_{r_0} \times \bar{B}_{r_0} \rightarrow \bar{B}_{r_0}$  ( $i = 1, 2$ ) defined by

$$T_i(x, y)(t) = f_i \left( t, x(\xi(t)), y(\xi(t)), \int_0^{\beta_i(t)} g_i(t, s, x(\eta(s)), y(\eta(s))) ds \right) \quad (22)$$

are well defined and satisfy condition (31) where  $G_i$  is given by (38) and

$$F_i(x, y)(t) = k \max\{x(t), y(t)\},$$

for  $i = 1, 2$ .

Using conditions (i)-(iv), for arbitrarily fixed  $t \in \mathbb{R}_+$  and  $i = 1, 2$  we get

$$\begin{aligned} & |T_i(x, y)(t)| \leq \\ & \leq \left| f_i \left( t, x(\xi_i(t)), y(\xi_i(t)), \int_0^{\beta_i(t)} g_i(t, s, x(\eta_i(s)), y(\eta_i(s))) ds \right) - f_i(t, 0, 0, 0) \right| \\ & \quad + |f_i(t, 0, 0, 0)| \\ & \leq k \max\{|x(\xi_i(t))|, |y(\xi_i(t))|\} + |f_i(t, 0, 0, 0)| \\ & + \Phi_i \left( m_i(t) \left| \int_0^{\beta_i(t)} g_i(t, s, x(\eta_i(s)), y(\eta_i(s))) ds \right| \right) \\ & \leq k \max\{\|x\|, \|y\|\} + M_i + \Phi_i(D), \end{aligned} \quad (24)$$

therefore,

$$\|T_i(x, y)\| \leq k \max\{\|x\|, \|y\|\} + M_i + \Phi_i(D). \quad (25)$$

Thus, from the estimate (48) we have  $T_i(\bar{B}_{r_0} \times \bar{B}_{r_0}) \subseteq \bar{B}_{r_0}$  for

$$r_0 = \max \left\{ \frac{M_1 + \Phi_1(D)}{1 - k}, \frac{M_2 + \Phi_2(D)}{1 - k} \right\}.$$

Next, by condition (ii) of Theorem (1.3.11), it is obvious that  $F_i$  and  $F_i(x)$  for  $x \in BC(\mathbb{R}_+)$  are continuous functions on  $BC(\mathbb{R}_+)$  and  $\mathbb{R}_+$ , respectively, and for

$i = 1, 2, x, y, u, v \in BC(\mathbb{R}_+)$  and  $t \in \mathbb{R}_+$  we have

$$\begin{aligned} & |T_i(x, y)(t) - T_i(u, v)(t)| = \\ & = \left| f_i \left( t, x(\xi_i(t)), y(\xi_i(t)), \int_0^{\beta_i(t)} g_i(t, s, x(\eta_i(s)), y(\eta_i(s))) ds \right) \right. \\ & \quad \left. - f_i \left( t, u(\xi_i(t)), v(\xi_i(t)), \int_0^{\beta_i(t)} g_i(t, s, u(\eta_i(s)), v(\eta_i(s))) ds \right) \right| \\ & \leq k \max\{|x(t) - u(t)|, |y(t) - v(t)|\} \end{aligned}$$

$$\begin{aligned}
 & +\Phi \left( m_i(t) \left| \int_0^{\beta_i(t)} g_i(t,s,x(\eta_i(s)),y(\eta_i(s))) ds \right. \right. \\
 & \quad \left. \left. - \int_0^{\beta_i(t)} g_i(t,s,u(\eta_i(s)),v(\eta_i(s))) ds \right| \right) \\
 & \leq |F_i(x,y)(t) - F_i(u,v)(t)| + \Phi(|G_i(x,y)(t) - G_i(u,v)(t)|) \\
 & \leq |F_i(x,y) - F_i(u,v)| + \Phi(\|G_i(x,y)(t) - G_i(u,v)(t)\|)
 \end{aligned}$$

therefore,

$$\|T_i(x,y) - T_i(u,v)\| \leq \|F_i(x,y) - F_i(u,v)\| + \Phi(\|G_i(x,y) - G_i(u,v)\|).$$

Obviously,  $F_i$  satisfies condition (10) and thus by Corollary (1.3.8), there exist  $x_0, y_0 \in BC(\mathbb{R}_+)$  that are solutions of the system of integral equations (1), and the proof is complete.

In the same way as the above proof, we can extend Theorem (3) for finite system of nonlinear integral equation

$$\begin{aligned}
 x_i(t) &= f_i(t, x_1(\xi_i(t)), \dots, x_n(\xi_i(t)), \\
 & \int_0^{\beta_i(t)} g_i(t,s,x_1(\eta_i(s)), \dots, x_n(\eta_i(s))) ds)
 \end{aligned}$$

where  $f_i, g_i, \xi_i, \eta_i$  and  $\beta_i$  satisfy certain conditions. As a corollary of Theorem (3) we have the main results of<sup>(17)</sup>.

**Corollary** <sup>(18)</sup>: Suppose that

(i)  $f: \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous and the function  $t \rightarrow f(t, 0, 0)$  is a member of the space  $BC(\mathbb{R}_+)$ ;

(ii) there exists  $k \in [0,1]$  such that

$$|f(t,x,y) - f(t,u,v)| \leq \frac{k}{2}(|x-v| + |y-v|), \tag{26}$$

for any  $t \geq 0$  and for all  $x, y, u, v \in \mathbb{R}$ ;

(iii) the functions  $\zeta, \eta, q: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  are continuous and  $\zeta(t) \rightarrow \infty$  as  $t \rightarrow \infty$ .

(iv)  $h: \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function and there exist  $x_0, y_0 \in \mathbb{R}$  and a positive constant  $d$  such that

$$\int_0^{q(t)} |h(t,s,r_0,y_0)| ds \leq d \tag{27}$$

for all  $t \in \mathbb{R}_+$ . In addition,

$$\lim_{t \rightarrow \infty} \int_0^{q(t)} \left| h(t, s, x(\eta(s)), y(\eta(s))) - h(t, s, u(\eta(s)), v(\eta(s))) \right| ds = 0 \quad (28)$$

$$\int_0^{q(t)} \left| h(t, s, x(\eta(s)), y(\eta(s))) - h(t, s, u(\eta(s)), v(\eta(s))) \right| ds \leq \infty \quad (29)$$

for any all  $t \in \mathbb{R}_+$  and uniformly respect to  $x, y, u, v \in BC \in (\mathbb{R}_+)$ :

Then the system of equations

$$\begin{cases} x(t) = f(t, x(\xi(t)), y(\xi(t))) + \int_0^{q(t)} h(t, s, x(\eta(s)), y(\eta(s))) ds, \\ y(t) = f(t, y(\xi(t)), x(\xi(t))) + \int_0^{q(t)} h(t, s, y(\eta(s)), x(\eta(s))) ds, \end{cases} \quad (30)$$

has at least one solution in the space  $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$ .

**Proof:** Take

$$f_1(t, x, y, z) = f(t, x, y) + z,$$

$$f_2(t, x, y, z) = f(t, y, x) + z,$$

$$g_1(t, s, x, y) = h(t, s, x, y),$$

$$g_2(t, s, x, y) = h(t, s, y, x),$$

in Theorem (1.3.11).

Now, we give an example where Theorem (3) can be applied but the previous results <sup>(19)</sup> are not applicable.

**Example (1.3.13)<sup>(20)</sup>:** Consider the system of integral equations

$$\begin{cases} x(t) = \frac{t^2(x(t) + y(t))}{2(1+t^4)} + \int_0^{t^2} \frac{s^2 \cos(sx(\sqrt{s})) + e^s (2 + \sin(x^4(\sqrt{x}) + y^4(\sqrt{s})))}{e^{t^2} (2 + \sin(x^4(\sqrt{x}) + y^4(\sqrt{s})))} ds \\ y(t) = \frac{\sin(t^2(x(t) + y(t)))}{2(1+t^4)} + \arctan \int_0^{\sqrt{t}} \frac{\sqrt{1+sy(s)} + ts^{11}(1+x^4(s) + y^4(s))}{(1+t^7)(1+x^4(t) + y^4(t))} ds \end{cases} \quad (5)$$

where  $t \in [0, \infty)$ .

Eq. (54) is a special case of Eq. (25) where

$$\zeta_1(t) = \zeta_2(t) = \eta_2(t) = t, \quad \beta_1(t) = t^2, \beta_2(t) = \eta_1(t) = \sqrt{s}$$

$$f_1(t, x, y, z) = \frac{t^2(x + y)}{2(1+t^4)} + z,$$

$$f_2(t, x, y, z) = \frac{\sin(t^2(x + y))}{2(1+t^4)} + \arctan z,$$

$$g_1(t, s, x, y) = \frac{s^3 \cos(sx) + e^s (2 + \sin(x^4 + y^4))}{e^{t^2} (2 + \sin(x^4 + y^4))}$$

$$g_2(t, s, x, y) = \frac{\sqrt[4]{1 + sy} + ts^{11} (1 + x^4 + y^4)}{(1 + t^7)(1 + x^4 + y^4)}$$

Now we check all conditions of Theorem (3). It is clear that condition (i) is satisfied. Assume that  $t \in \mathbb{R}_+$  and  $x, y, z, u, v, w \in \mathbb{R}$ . Then we get

$$\begin{aligned} |f_1(t, x, y, z) - f_1(t, u, v, w)| &\leq \frac{t^2}{1 + t^4} \frac{|x - u| + |y - v|}{2} + |z - w| \\ &\leq \frac{1}{2} \max\{|x - u| + |y - v|\} + |z - w| \end{aligned}$$

and

$$\begin{aligned} |f_2(t, x, y, z) - f_2(t, u, v, w)| &\leq \frac{|\sin(t^2(x - u + y - v))|}{2(1 + t^4)} \\ &\quad + |\arctan(z) - \arctan(w)| \\ &\leq \frac{1}{2} \max\{|x - u|, |y - v|\} + |z - w| \end{aligned}$$

Therefore  $f_1$  and  $f_2$  satisfy condition (ii) of Theorem (1.3.11) with  $k = \frac{1}{2}$ . Also it is clear that  $f_i$  and  $g_i$  are continuous and by simple calculation we obtain that

$$M_1 = \sup \left\{ \frac{t^2(0 + 0)}{2(1 + t^4)} + 0 : t \in \mathbb{R}_+ \right\} = 0,$$

$$M_2 = \sup \left\{ \frac{\sin(t^2(0 + 0))}{2(1 + t^4)} + 0 : t \in \mathbb{R}_+ \right\} = 0$$

$$\left| \frac{s^3 \cos(sx(\sqrt{s})) + e^s (2 + \sin(x^4(\sqrt{s}) + y^4(\sqrt{s})))}{e^{t^2} (2 + \sin(x^4(\sqrt{s}) + y^4(\sqrt{s})))} \right| \leq \left| \frac{s^3 + 2e^s}{e^{t^2}} \right|$$

$$\lim_{t \rightarrow \infty} \left| \int_0^{\sqrt{t}} \frac{\sqrt{1 + sx(s)} + ts^{11} (1 + x^4(t) + y^4(s))}{(1 + t^7)(1 + x^4(t) + y^4(s))} ds \right| = \frac{1}{12}$$

$$\left| g_1(t, s, x(\eta_1(s)), y(\eta_1(s))) - g_1(t, s, u(\eta_1(s)), v(\eta_1(s))) \right| \leq \frac{2s^3}{e^{t^2}}$$

$$\left| g_2(t, s, x(\eta_2(s)), y(\eta_2(s))) - g_2(t, s, u(\eta_2(s)), v(\eta_2(s))) \right| \leq \frac{2(1 + s)}{1 + t^7}$$

Thus,  $D \leq \infty$  and we have

$$\lim_{t \rightarrow \infty} \left| \int_0^{\beta_t(t)} g_t(t, s, x(\eta_t(s)), y(\eta_t(s))) - g_t(t, s, u(\eta_t(s)), v(\eta_t(s))) ds \right| = 0$$

Therefore, as a result of Theorem (3), the system of integral equations (30) has at least one solution in the space  $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$ .

### 5.1 Conclusion:

The concept of measure of non-compactness has played a basic role in nonlinear functional analysis, especially in metric and topological fixed point theory. Up to now, several papers have been published on the existence and behavior of solutions of nonlinear differential and integral equations, using the technique of measure of non-compactness. The aim is to obtain an extension of Darbo's fixed point theorem and to present some results on the existence of coupled fixed points for a class of condensing operators in Banach spaces (i.e. mappings under which the image of any set is in a certain sense more compact than the set itself). Moreover, as an application, we study the problem of the existence of solutions for a general system of nonlinear integral equations. Here, we give some notations, definitions and auxiliary facts.

The first measure of non-compactness was introduced by Kuratowski in the following way.

$$\alpha(S) = \inf\{\delta > 0 \mid S = \bigcup_{i=1}^n S_i \text{ for some } S_i \text{ with } \text{diam}(S_i) \leq \delta \text{ for } 1 \leq i \leq n < \infty\}$$

These measures share several useful and characterizing properties. In many classical texts, this concept has been defined axiomatically to emulate some of important common properties of the measures  $\chi$  and  $\alpha$ , but here we use the definition given.

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